Appendix: Basic Notions

A

Introduction

In this appendix, we set out the basic definitions and axioms that form the foundation for the work we do in the rest of the book. Many of the notions presented here are intuitive, and we merely formalize them by stating them as axioms or theorems. We begin by taking the terms *point*, *line*, *plane*, and *space* as undefined (not in the sense that we don't know what they are, but in the sense that they have not been given formal definitions). We also take as undefined the concepts of *set*, *belongs to*, or *is an element of* a set. In the formal axiomatic approach that we will be taking, we use only the properties of the undefined terms that we state in the axioms—we are not allowed to make conclusions based on drawings. Our first axiom deals with lines, planes, and space as sets of points.

Axiom A.1 *Lines, planes, and space are sets of points. Space contains all points.*

In geometry, it is common to use the terms *on*, *in*, *passes through*, and *lies on*. We say that a point is *on* a line rather than *belongs* to a line. Synonymously, we say that a line *passes through* a given point. We also say that a point is *on* a plane or *in* a plane. A line *lies* in a plane if it is a subset of the plane—that is, if every point on the line is also in the plane. Points that are on the same line are called **collinear**. Points that are on the same plane are **coplanar**.

The following axioms describe the fundamental relationships among points, lines, and planes. The accompanying figures are merely models for the relationships; they do not represent the only possible configuration.

Axiom A.2 *Any two distinct points are on exactly one line. Every line contains at least two points.*

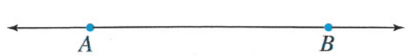

Figure A.1

Remark It may seem that Axiom A.2 implies that all lines are straight; otherwise, more than one line could be drawn through two points. First, notice that "straightness" has not yet been defined. Second, it is possible to show an example of geometry in which lines are objects that satisfy Axiom A.2 and other axioms in this section but are not "straight."

Axiom A.3 *Any three noncollinear points are on exactly one plane. Each plane contains at least three noncollinear points.*

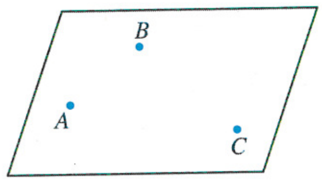

Figure A.2

Axiom A.4 *If two points of a line are in a plane, then the entire line is in the plane.*

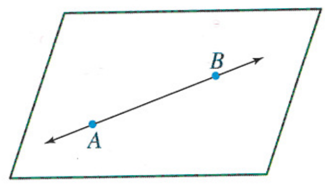

Figure A.3

Axiom A.5 *In space, if two planes have a point in common, then the planes have an entire line in common.*

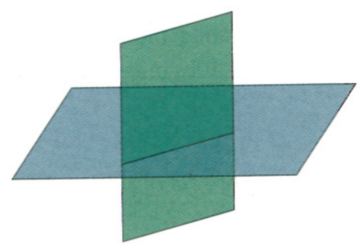

Figure A.4

Axiom A.6 *In space, there exist at least four points that are noncoplanar.*

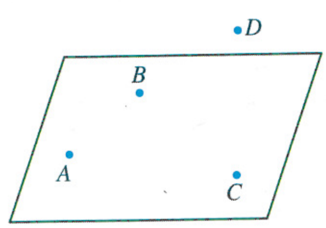

Figure A.5

Remarks

- Axiom A.2 is often encountered in an equivalent form: "Two points determine a unique line" or "There is one and only one line passing through two distinct points."

- Similarly, an equivalent form of Axiom A.3 is "Three noncollinear points determine a unique plane."
- Notice the second sentence in Axioms A.1 and A.2. We know intuitively that lines and planes contain infinitely many points, but this fact does not follow from the preceding axioms. Additional axioms will be needed to assure infinitude of points on a line.

A.1 Notation for Points, Lines, and Planes

It is customary to designate points by capital letters of the Latin alphabet. Axiom A.2 assures that a line can be named by any two points on it. The line containing points A and B, as in Figure A.1, will be denoted by $\overleftrightarrow{AB}$ or line AB. Whenever convenient, we also may name a line by a single letter. In this text, only lowercase letters from the Latin alphabet are used to name lines.

Axiom A.3 assures that a plane can be named by any three noncollinear points on the plane. For example, the plane containing the upper face $ABCD$ of the box in Figure A.6 can be named in each of the following ways: plane ABC, plane BCD, plane ADC, and plane ABD. (Of course, the plane can also be named by any three other noncollinear points not labeled in the figure.) Whenever convenient, we may also name a plane by a single letter. In this text, only lowercase Greek letters will be used to name planes.

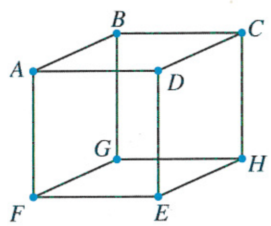

Figure A.6

Recall that the intersection of two sets is the set of all elements that are common to both sets. We know intuitively that two distinct lines either do not intersect (have no points in common) or intersect in exactly one point. This understanding leads to our first theorem.

Theorem A.1

If two distinct lines intersect, they intersect in exactly one point.

Proof

Let the lines be k and ℓ. It is given that the lines intersect. Therefore, there exists a point P that is on both lines. We want to show that k and ℓ have no other points in common. For that purpose, we use an indirect proof. Suppose there is another point Q on k and ℓ, as in Figure A.7.

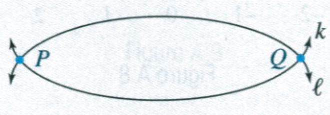

Figure A.7

By Axiom A.2, there is a unique line through P and Q. Hence, $k = \ell$, which contradicts the hypothesis that the lines are different. Consequently, the existence of another intersection point Q must be rejected. □

Axiom A.3 assured us that three noncollinear points determine a plane. The thoughtful reader might set his or her mind to work to see if he or she could come up with another way of uniquely determining a plane. In fact, you may have come up with another way already—namely, "two parallel lines uniquely determine a plane." Before the concept has any meaning in our system, however, we must formally define parallel lines.

Definition of Parallel and Skew Lines Two lines are parallel if they lie in the same plane and do not intersect. Lines that do not intersect and are not contained in any single plane are called skew lines.

If ℓ and m are parallel, we write $\ell \parallel m$. In Figure A.6, for example, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $\overleftrightarrow{AB} \parallel \overleftrightarrow{EH}$, and $\overleftrightarrow{AC} \parallel \overleftrightarrow{FH}$, but $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ are skew lines.

Now Solve This A.1

1. What is the maximum number of intersection points determined by n lines in the same plane?
2. What is the maximum number of lines determined by n points? Does it matter if the points are in the same plane?

A.2 Intuitive Background for the Coordinate System and Distance

Following G. D. Birkhoff's (1884–1944) axioms, we now introduce the concept of distance, assuming the existence and properties of real numbers. We start with an intuitive background that will motivate the axioms and definitions that follow.

Historical Note: George David Birkhoff

George David Birkhoff was one of the most distinguished leaders in American mathematics and the preeminent U.S. mathematician of his time. He taught at Harvard from 1912 until his death in 1944. Birkhoff proposed an axiom system for Euclidean geometry in his 1941 text *Basic Geometry*.

Given any line, a **coordinate system** on the line can be created by choosing an arbitrary point O on the line and having that point correspond to 0. This point O is called the **origin.** Next to the right of point O, another point P is chosen (see Figure A.8) to which corresponds the number 1. The segment $\overline{OP}$ is called a **unit** segment.

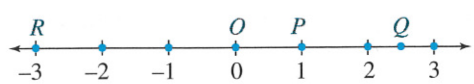

Figure A.8

By marking off segments equal to the length of OP repeatedly to the left and right of O, we find points corresponding to the integers. By dividing segments into an appropriate number of equal parts, we find points that correspond to all rational numbers. You most likely know that any real number (and not only rational numbers) corresponds to some point on the line and, conversely, that every point on the line corresponds to some real number. (In Chapter 3, we have shown how to find points that correspond to real numbers such as $\sqrt{2}$ and $\sqrt{5}$.) Thus there is one-to-one correspondence between the points on a line and the real numbers. Such a correspondence is called a **coordinate system** for a line. The number corresponding to a given point P is called the **coordinate** of P. Thus the coordinate of Q in Figure A.8 is 2.5.

If we denote the line in Figure A.8 by x, we write the coordinate of Q as x_Q. Thus $x_Q = 2.5$ and $x_R = -3$. We can find the distance between two points by using the coordinates of the points. For example, in Figure A.8 we have $PQ = OQ - OP = x_Q - x_P = 2.5 - 1 = 1.5$. We can also find RP by finding the difference between the coordinates of the points: $RP = x_P - x_R = 1 - (-3) = 4$. Because distance is a non-negative number and it is cumbersome to indicate which point has the greater coordinate, we use the absolute value function. Thus $AB = |x_A - x_B|$.

Based on this discussion we introduce the following axiom and definitions.

Axiom A.7 **The Ruler Postulate** *The points on a line can be put in one-to-one correspondence with the real numbers.*

Notice that this axiom implies that every line has an infinite number of points.

Definition of a Coordinate System for a Line The correspondence in Axiom A.7 is called a coordinate system for a line. A line with a coordinate system is called a number line.

Definition of a Distance Between Two Points The distance between points A and B, denoted by AB, is the real number $|x_A - x_B|$, where x_A and x_B are the coordinates of A and B, respectively, in a coordinate system for $\overleftrightarrow{AB}$.

You may have already observed that the distance between two points depends on the unit chosen for the coordinate system. If Q and R stay in the same place but we change the position of the point that corresponds to the number 1, then QR will change as well. Also, because the distance from a point A to the origin is the real number $|x_A|$, and there exist real numbers as large as we might wish, we can conclude that there are points on a line as far from the origin as we wish. Therefore, we can say that a line is infinite in length.

We can use the concept of distance to define what we mean when we say that a point is between two other points.

Definition of Betweenness B is between A and C if and only if A, B, and C are collinear and $AB + BC = AC$ (see Figure A.9). In this case we write A–B–C.

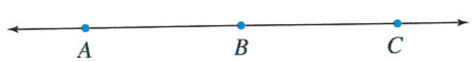

Figure A.9

Using the concept of betweenness for points, it is possible to define various geometric figures.

Definition of a Segment The segment $\overline{AB}$ consists of the points A and B and all the points between A and B.

The length of segment $\overline{AB}$ is the distance between A and B, denoted by AB. A point M between A and B such that $AM = MB$ is a **midpoint** of $\overline{AB}$. Our intuition tells us that every segment has exactly one midpoint. We can prove this fact by using Axiom A.7 and our definitions. You will be guided in the process of finding a proof in the problem set at the end of this appendix. Most of us probably find it more satisfying to prove statements that are not intuitively obvious. Nevertheless, you may find the challenge of proving a statement rewarding in itself even if the statement seems obvious. We also want you to realize that even intuitively obvious statements can be logically deduced from the axioms and definitions.

Example A.1 Given two points A and B on a number line with coordinates x_A and x_B, respectively, find the coordinate x_M of M, the midpoint of $\overline{AB}$. (See Figure A.10.)

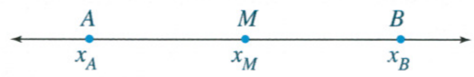

Figure A.10

Solution

Assume $x_B > x_A$. The definition of the midpoint implies that $AM = MB$. This equation implies

$$x_M - x_A = x_B - x_M$$

$$2x_M = x_A + x_B$$

$$x_M = \frac{(x_A + x_B)}{2}$$

Now Solve This A.2

A student approaches the solution of Example A.1 as follows: Because $AB = x_B - x_A$ and the midpoint of AB is halfway between A and B, the coordinate of the midpoint should be $\frac{1}{2}(x_B - x_A)$. The student realizes the answer is wrong but would like to know why and how to use her approach to obtain the correct answer. How would you respond?

Definition of a Ray The ray $\overrightarrow{AB}$ (shown in Figure A.11a) is the union of $\overline{AB}$ and the set of all points C such that B is between A and C. The point A is called the endpoint of the ray. The rays having a common endpoint and whose union is a straight line are opposite rays. In Figure A.11b, $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are opposite rays.

Figure A.11

Definition of an Angle An angle is a union of two rays with a common endpoint. The common endpoint is the vertex of the angle, and the two rays are called the sides of the angle.

In Figure A.12a, the angle shown is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and its vertex is A. (Using set notation, the angle is $\overrightarrow{AB} \cup \overrightarrow{AC}$.) The angle in Figure A.12a is designated by $\angle BAC$ or $\angle CAB$. When there is no danger of ambiguity, it is common practice to name an angle by its vertex. Thus the angle in Figure A.12a can also be denoted by $\angle A$. If the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are on the same line—that is, if A, B, and C are collinear, as in Figure A.12b—then $\angle ABC$ is called a **straight angle**.

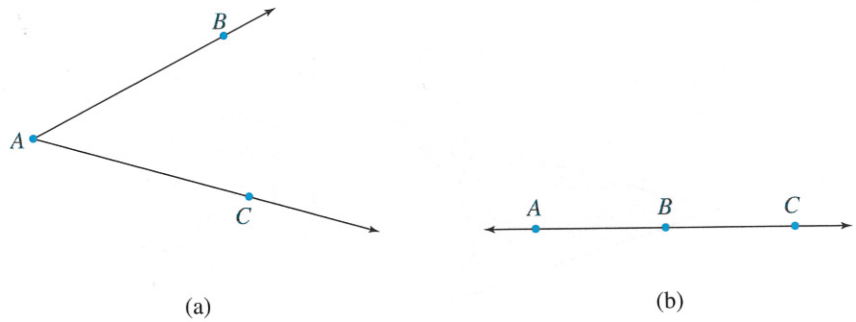

Figure A.12

If A, B, and C are three noncollinear points, then the union of the three segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called a **triangle** and is denoted by $\triangle ABC$ (shown in Figure A.13). The three segments are called the **sides** of the triangle, and the three points are called the **vertices**.

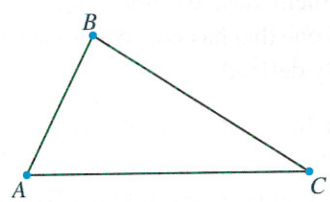

Figure A.13

At this juncture, we could prove several statements that are intuitively obvious. For example, if A–B–C (B is between A and C), then C–B–A (B is between C and A). Also, $AB = BA$ and $\triangle ABC = \triangle CBA$. By "equal," we mean "exactly" equal in the set theory sense: Two sets are equal if they contain the same elements. To give a rigorous treatment of the next topics we will discuss, we need the concept and properties of half planes. Intuitively, we know that any line divides the plane into two parts separated by the line and that each part is referred to as the **half plane**. This fact will be introduced in Axiom A.8, the Plane-Separation Axiom. Before we get to this axiom, however, it will be useful to define what we mean by a convex set.

Definition of a Convex Set A set is convex if for every two points P and Q belonging to the set, the entire segment $\overline{PQ}$ is in the set.

Notice that the interiors of the triangle as well as the circle in Figure A.14 are convex sets. (We are making this statement on an intuitive basis; the interior of a triangle has not been defined yet.) Also, the segment $\overline{AB}$ is a convex set. However, the interiors of the figures in Figure A.15 are not convex, as the segment $\overline{PQ}$ in each figure does not entirely belong to the figure.

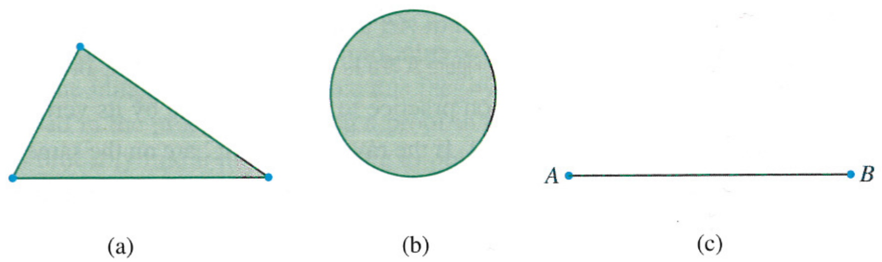

Figure A.14

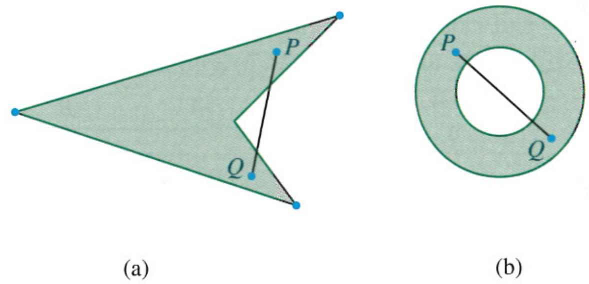

Figure A.15

Problems Solved and Unsolved

Properties of convex sets have been extensively investigated, and the concept of convex sets has important applications in mathematics. We briefly describe here two properties of convex sets, one that has been proved and one that has not. We use terminology that should be intuitively clear but has not yet been precisely defined.

Chords intersecting at 60°: In the interior of any closed convex curve (see Figure A.16), there exists a point P and three chords through P (a chord of a region is a segment connecting any two boundary points of the region) with the following property: The six angles formed at P are 60° each and P is the midpoint of each of the chords. (The proof of this statement requires more extensive study of a convex sets. Figure A.16 illustrates the statement for a circle.)

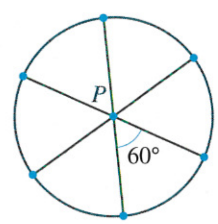

Figure A.16

Equichordal points: A point P in the interior of a region is an equichordal point if all chords through P are of the same length. For example, the center of a circle is an equichordal point of the circle. However, there exist noncircular regions that have an equichordal point. In 1916, Fujiwara raised the question of whether there exists a plane convex region that has two equichordal points. No one has been able to give a complete answer to this question. In 1984, Spaltenstein described a construction of a convex region on a sphere that has two equichordal points. (This does not answer the question posed in 1916, as the region is on a sphere and hence is not planar.)

Axiom A.8 **The Plane-Separation Axiom** *Each line in a plane separates all the points of the plane that are not on the line into two nonempty sets, called the half planes, with the following properties:*

1. *The half planes are disjoint (have no points in common) convex sets.*
2. *If P is in one half plane and Q is in the other half plane, the segment $\overline{PQ}$ intersects the line that separates the plane.*

Notice that neither of the half planes in Axiom A.8 includes the line. Thus a line divides the plane into three mutually disjoint subsets: the two half planes and the line. Also, it follows from Axiom A.8 that a half plane is determined by a line and a point not on the line. Thus we can refer to the two half planes in Figure A.17 as "the half plane of ℓ containing P" and "the half plane of ℓ containing Q."

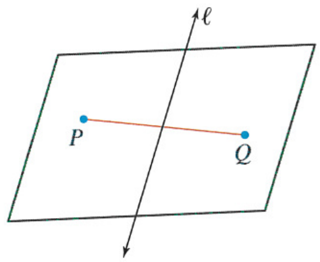

Figure A.17

Using Axiom A.8, it is possible to prove a theorem named after the German mathematician Moritz Pasch (1843–1930), which states that if a line intersects one side of a triangle and does not go through any of its vertices, it must also intersect another side of the triangle (see Figure A.18). In 1882, Pasch published one of the first rigorous treatises on geometry where he stated this theorem as an axiom (he did not use Axiom A.8 as an axiom). Pasch realized that Euclid often relied on assumptions made visually from diagrams and contributed to filling the gaps in Euclid's reasoning.

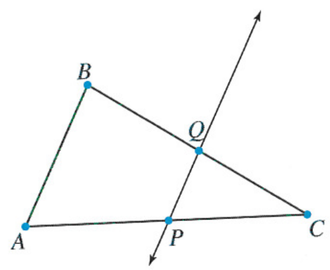

Figure A.18

We state Pasch's Axiom as a theorem and leave its proof for you to explore in the problem set at the end of this appendix.

Theorem A.2

Pasch's Axiom

If a line intersects a side of a triangle and does not intersect any of the vertices, it also intersects another side of the triangle.

Using Axiom A.8, it becomes possible to precisely define the interior of an angle and hence the interior of a triangle.

Definition of the Interior of an Angle If A, B, and C are not collinear, then the interior of $\angle BAC$ is the intersection of the half plane of $\overleftrightarrow{AB}$ containing C with the half plane of $\overleftrightarrow{AC}$ containing B. (See Figure A.19.)

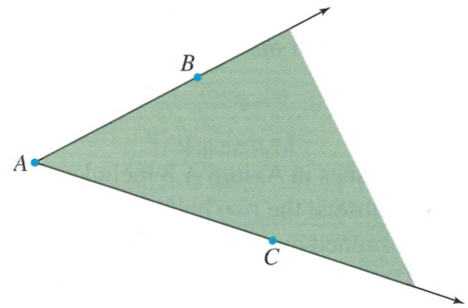

Figure A.19

Remark This definition is valid (or, as it is commonly referred to in mathematics, **well defined**) if it is independent of the choices of B and C on the sides of the angle. It can be proved that this indeed is the case.

Using the preceding definition, we can define the interior of a triangle. Coming up with an appropriate definition is left to you as an exercise. The definition of the interior of an angle can also be used to define **betweenness** for rays.

Definition of Betweenness of Rays $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and if and only if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is in the interior of $\angle BAC$. (See Figure A.20.)

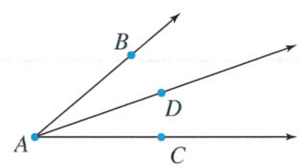

Figure A.20

We can use this definition to prove the following visually obvious theorem. (Its proof is also left as an exercise for you.)

Theorem A.3

$\overrightarrow{CD}$ intersects side $\overline{AB}$ of $\triangle ABC$ between A and B if and only if $\overrightarrow{CD}$ is between $\overrightarrow{CA}$ and $\overrightarrow{CB}$. (See Figure A.21.*)*

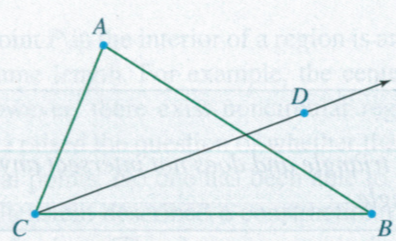

Figure A.21

A.3 Angle Measurement

Angles are commonly measured in degrees with a protractor. (Another unit of measurement for angles is the **radian**.) To measure $\angle EAB$, we place the protractor as shown in Figure A.22 and read off the measure of the angle as 140°.

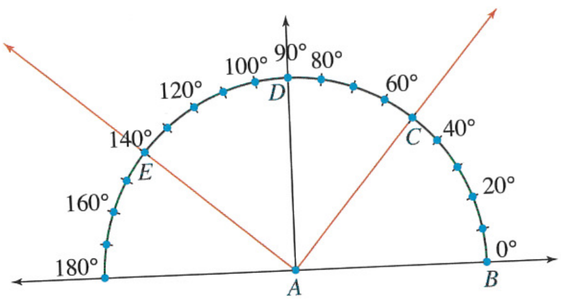

Figure A.22

The following axioms and definitions formalize our intuitive knowledge of the protractor.

Axiom A.9 **The Angle Measurement Axiom** *There is a real number between 0 and 180 that corresponds to every angle $\angle BAC$. The number 180 corresponds to the straight line.*

The number in Axiom A.9 is called the **degree measure of the angle** and is written $m(\angle BAC)$. In Figure A.22, $m(\angle BAC) = 50°$ and we say that $\angle BAC$ is a 50-degree angle, written as 50°.

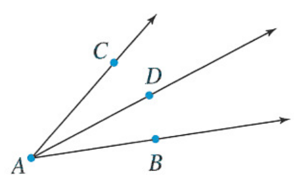

Figure A.23

Axiom A.10 **The Angle Construction Postulate** *Let $\overrightarrow{AB}$ be a ray on the edge of a half plane. For every real number r, $0 < r < 180$, there is exactly one ray, with C in the half plane, such that $m(\angle CAB) = r$.*

Axiom A.11 **The Angle Addition Postulate** *If D is a point in the interior of $\angle BAC$ (see Figure A.23), then $m(\angle BAD) + m(\angle DAC) = m(\angle BAC)$.*

Notice that Axiom A.11 implies that measures of angles can be computed by subtraction. For example, $m(\angle BAD) = m(\angle BAC) - m(\angle DAC)$.

Problem Set

In the following problems, use the axioms, definitions, and theorems presented in this appendix to prove what is required.

•**1.** Prove that

a. If two lines intersect, they lie in exactly one plane.

b. Two parallel lines determine a unique plane.

2. Consider the undefined terms: *ball*, *player*, and *belongs to*. Also consider the following axioms:

Axiom 1 There is at least one player.

Axiom 2 To every player belong two balls.

Axiom 3 Every ball belongs to three players.

Prove or disprove each of the following:

a. There are at least two balls.

b. There are at least three players.

c. There is always an even number of balls.

d. The number of players is always odd.

(*Hint:* You may want to model the balls by a point and players by segments.)

3. Consider the undefined terms: *line*, *point*, and *belongs to*. Also consider the following axioms:

Axiom 1 Any two lines have exactly one point in common.

Axiom 2 Every point is on (belongs to) exactly two lines.

Axiom 3 There are exactly four lines.

Answer each of the following:

a. How many points are there? Prove your answer.

b. Prove that there are exactly three points on each line.

(*Hint:* You may prefer to substitute the term *point* by *person* and *line* by *committee*.)

4. Using the terminology presented in this appendix, come up with a precise definition of the interior of a triangle.

5. Is the *betweenness* of *rays* well defined? Explain.

•**6.** Which of the following is true *always*, *sometimes* but not *always*, or *never?* Justify your answers.

•**a.** The intersection of two convex sets is convex.

b. The intersection of two nonconvex sets is a nonconvex set.

•**c.** The union of two convex sets is convex.

d. The union of two nonconvex sets is nonconvex.

7. Prove Pasch's Axiom.