41–42 ▪ Reconsider the data describing the levels of a medication in the blood of two patients over the course of several days (measured in milligrams per liter), used in Section 1.5, Exercises 53 and 54.

Day	Medication Level in Patient 1 (mg/L)	Medication Level in Patient 2 (mg/L)
0	20.0	0.0
1	16.0	2.0
2	13.0	3.2
3	10.75	3.92

41. For the first patient, graph the updating function, and cobweb starting from the initial condition on day 0. Find the equilibrium.

42. For the second patient, graph the updating function, and cobweb starting from the initial condition on day 0. Find the equilibrium.

43–44 ▪ Cobweb and find the equilibrium of the following discrete-time dynamical systems.

43. Consider a bacterial population that doubles every hour, but 1.0×10^6 individuals are removed after reproduction (Section 1.5, Exercise 57). Cobweb starting from $b_0 = 3.0 \times 10^6$ bacteria. Is the result consistent with the result of Exercise 57?

44. Consider a bacterial population that doubles every hour, but 1.0×10^6 individuals are removed before reproduction (Section 1.5, Exercise 58). Cobweb starting from $b_0 = 3.0 \times 10^6$ bacteria. Is the result consistent with the result of Exercise 58?

45–46 ▪ Consider the following general models for bacterial populations with harvest.

45. Consider a bacterial population that doubles every hour, but h individuals are removed after reproduction. Find the equilibrium. Does it make sense?

46. Consider a bacterial population that increases by a factor of r every hour, but 1.0×10^6 individuals are removed after reproduction. Find the equilibrium. What values of r produce a positive equilibrium?

Computer Exercises

47. Use your computer (it may have a special feature for this) to find and graph the first 10 points on the solutions of the following discrete-time dynamical systems. The first two describe populations with reproduction and immigration of 100 individuals per generation, and the last two describe populations that have 100 individuals harvested or removed each generation.

a. $b_{t+1} = 0.5b_t + 100$ starting from $b_0 = 100$

b. $b_{t+1} = 1.5b_t + 100$ starting from $b_0 = 100$

c. $b_{t+1} = 1.5b_t - 100$ starting from $b_0 = 201$

d. $b_{t+1} = 1.5b_t - 100$ starting from $b_0 = 199$

e. What happens if you run the last one (part d) for 15 steps? What is wrong with the model?

48. Compose the medication discrete-time dynamical system $M_{t+1} = 0.5M_t + 1.0$ with itself 10 times. Plot the resulting function. Use this composition to find the concentration after 10 days, starting from concentrations of 1.0, 5.0, and 18.0 milligrams per liter. If the goal is to reach a stable concentration of 2.0 mg/l, do you think this is a good therapy?

1.7 Expressing Solutions with Exponential Functions

The solution associated with the bacterial discrete-time dynamical system given by $b_{t+1} = 2.0b_t$ is

$$b_t = 2.0^t$$

when $b_0 = 1.0$. As a function of t, the solution is an example of an **exponential function.** To find how long it will take the population to reach 100 requires solving an equation where the variable t appears in the exponent. Solving for t requires converting this function into a standard form with the base e and working with the inverse of the exponential function, the **natural logarithm.** We will study the **laws of exponents** and the **laws of logarithms.** More generally, what happens to the discrete-time dynamical system and solution if some of the bacteria die during the course of each hour? We will see that the solution is again an exponential function, with base equal to the **per capita production** of the bacteria.

Bacterial Population Growth in General

The bacteria studied hitherto have doubled in number each hour. Each bacterium divided once and both "daughter" bacteria survived. Suppose instead that only a fraction σ (sigma) of the daughters survive. Instead of 2.0 offspring per bacteria, we find an average of 2σ offspring (Figure 1.7.100). For example, if only 75% of offspring survived

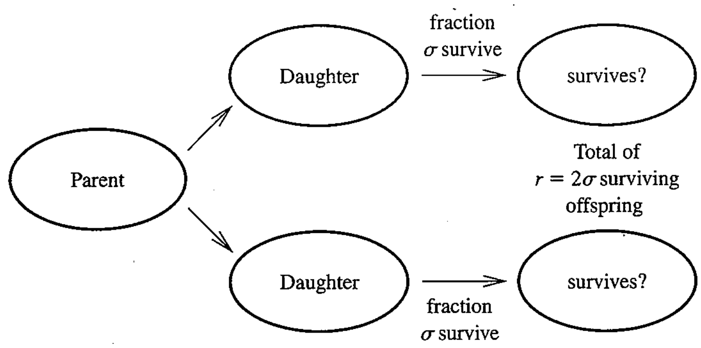


FIGURE 1.7.100
Bacterial population growth with reproduction and mortality

($\sigma = 0.75$), there are an average of only 1.5 surviving offspring per parent. Let

$$r = 2\sigma$$

The new parameter r represents the number of new bacteria produced per bacterium and is called the **per capita production.**

In terms of the parameter r, the discrete-time dynamical system is

$$b_{t+1} = rb_t$$

This fundamental equation of population biology says that the population at time $t+1$ is equal to the per capita production (the number of new bacteria per old bacterium) times the population at time t (the number of old bacteria), or

$$\text{new population} = \text{per capita production} \times \text{old population}$$

Example 1.7.1 Discrete-time Dynamical System if Most Offspring Survive

If $\sigma = 0.75$, then $r = 2 \cdot 0.75 = 1.5$. The discrete-time dynamical system is

$$b_{t+1} = 1.5b_t$$

If $b_0 = 100$, then $b_1 = 1.5 \cdot 100 = 150$. The population increases by 50% each hour.

Example 1.7.2 Discrete-time Dynamical System if Few Offspring Survive

If $\sigma = 0.25$, then $r = 2 \cdot 0.25 = 0.5$. The discrete-time dynamical system is

$$b_{t+1} = 0.5b_t$$

If $b_0 = 100$, then $b_1 = 0.5 \cdot 100 = 50$. Because the value of the survival σ is so small, this population decreases by 50% each hour.

Starting from a population with b_0 bacteria, we can apply the discrete-time dynamical system repeatedly to derive a solution, much as we did in Example 1.5.11 with the particular value $r = 2$ (Figure 1.7.101). We find

$$\begin{aligned} b_1 &= rb_0 \\ b_2 &= rb_1 = r^2 b_0 \\ b_3 &= rb_2 = r^3 b_0 \end{aligned}$$

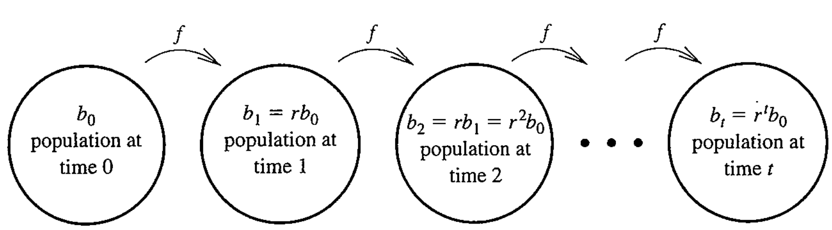


FIGURE 1.7.101
Bacterial population growth

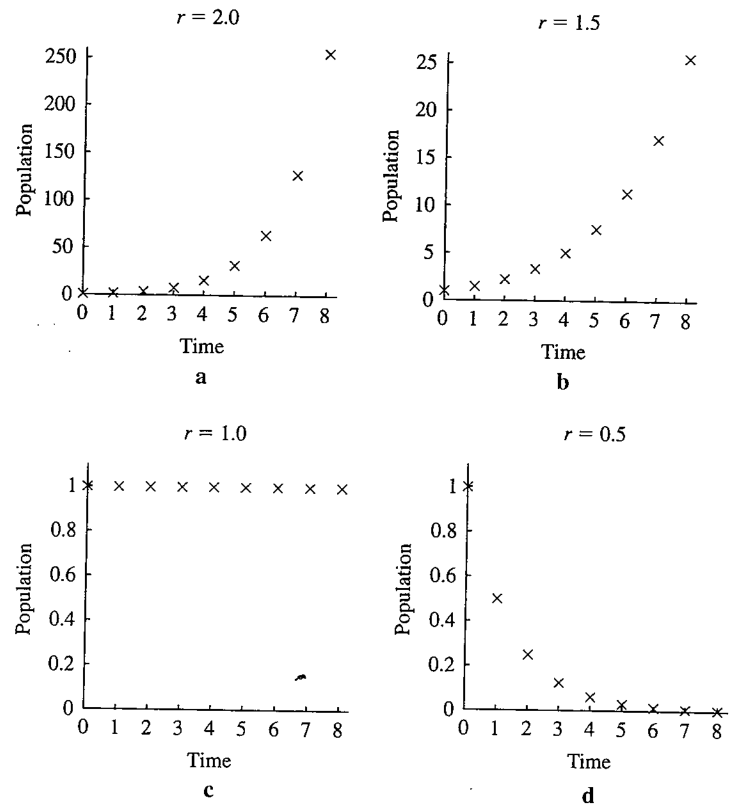


FIGURE 1.7.102
Growing and declining bacterial populations

Each hour, the initial population b_0 is multiplied by the per capita production r. After t hours, the initial population b_0 has been multiplied by t factors of r. Therefore,

$$b_t = r^t b_0$$

How do these solutions behave for different values of the per capita production r? Results with four values of r starting from $b_0 = 1.0$ are given in the following table.

t	$r = 2.0$	$r = 1.5$	$r = 1.0$	$r = 0.5$
0	1.0	1.0	1.0	1.0
1	2.0	1.5	1.0	0.5
2	4.0	2.25	1.0	0.25
3	8.0	3.37	1.0	0.125
4	16.0	5.06	1.0	0.0625
5	32.0	7.59	1.0	0.0312
6	64.0	11.4	1.0	0.0156
7	128.0	17.1	1.0	0.00781
8	256.0	25.6	1.0	0.00391

In the first two columns, $r > 1$ and the population increases each hour (Figure 1.7.102a and b). In the third column, $r = 1$ and the population remains the same hour after hour (Figure 1.7.102c). In the final column, $r < 1$ and the population decreases each hour (Figure 1.7.102d). We summarize these observations in the following table.

Value of r	Behavior of Population
$r > 1$	population increases
$r = 1$	population remains constant
$r < 1$	population decreases

A population with $r = 1$ exactly replaces itself each generation and retains a constant size, even though the individuals in the population change. This is consistent with our finding that any value of b is an equilibrium when $r = 1$ (Example 1.6.7).

Laws of Exponents and Logs

In the solution $b_t = r^t b_0$, the variable t appears in the exponent, in contrast to a function such as $f(t) = t^3$ where the variable t is raised to a power. For any positive number a, the **exponential function to the base a** is written

$$f(x) = a^x$$

and is read "a to the xth power." This function takes x as input and returns x factors of a multiplied together. The notation generalizes that used in equations such as

$$a^2 = a \cdot a$$

The key to using exponential functions is knowing the **laws of exponents**, summarized in the following table. This table also includes examples using $a = 2$ that can help in remembering when to add and when to multiply.

Laws of exponents

	General Formula	Example with $a=2$, $x=2$, and $y=3$
Law 1	$a^x \cdot a^y = a^{x+y}$	$2^2 \cdot 2^3 = 2^5 = 32$
Law 2	$(a^x)^y = a^{xy}$	$(2^2)^3 = 2^6 = 64$
Law 3	$a^{-x} = \frac{1}{a^x}$	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
Law 4	$\frac{a^y}{a^x} = a^{y-x}$	$\frac{2^3}{2^2} = 2^{3-2} = 2$
Law 5	$a^1 = a$	$2^1 = 2$
Law 6	$a^0 = 1$	$2^0 = 1$

The exponential function is defined for all values of x, including negative numbers and fractions. What does it mean to multiply half an a or -3 a's together? These expressions must be computed with the laws of exponents.

Example 1.7.3 Negative Powers

To compute a^{-3}, apply law 3 to find

$$a^{-3} = \frac{1}{a^3}$$

For example,

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8} = 0.125$$

Negative powers in the numerator are positive in the denominator.

Example 1.7.4 Fractional Powers

To compute $a^{0.5}$, we raise this unknown quantity to the 2nd power (square it) and use law 2 to find

$$\left(a^{0.5}\right)^2 = a^{0.5 \cdot 2} = a^1 = a$$

Therefore, a to the 0.5 power is the number that, when squared, gives back a. In other words, a to the 0.5 power is the square root of a. For example

$$2^{0.5} = \sqrt{2} \approx 1.41421$$

For reasons that will make sense only with a bit of calculus (Section 2.8), the base most commonly used throughout the sciences is the irrational number

$$e = 2.718281828459\ldots.$$

The function

$$f(x) = e^x$$

which is read "e to the x," is called the **exponential function** to the base e, or simply the **exponential function** (Figure 1.7.103). Calculators and computers often abbreviate this as exp. The domain of this function consists of all numbers, and the range is all **positive** numbers.

Example 1.7.5 The Laws of Exponents for the Base e

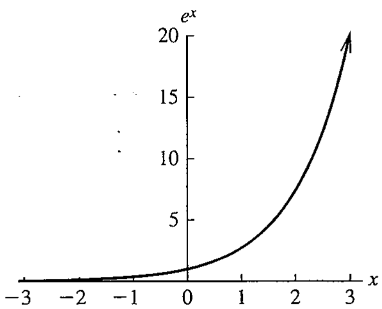


FIGURE 1.7.103
Graph of the exponential function

- $e^3 \cdot e^4 = e^{3+4} = e^7$ (law 1).
- $e^3 + e^4$ cannot be simplified with a law of exponents.
- $(e^3)^4 = e^{3\cdot 4} = e^{12}$ (law 2).
- $e^{-2} = \dfrac{1}{e^2}$ (law 3).
- $\dfrac{e^4}{e^3} = e^{4-3} = e^1 = e$ (laws 4 and 5).
- $e^0 = 1$ (law 6).

The graph of the exponential function crosses every positive horizontal line only once and thus passes the horizontal line test for having an inverse (see "Finding Inverse-Functions," Section 1.2, p. 17). The inverse is the natural log.

Definition 1.12 The inverse function of the exponential function e^x is called the **natural logarithm** (or natural log). The natural log of x is written $\ln(x)$. The natural logarithm has a domain consisting of all positive numbers.

From the definition of the inverse (Definition 1.6),

$$\ln(e^x) = x$$
$$e^{\ln(x)} = x$$

The graph of the natural logarithm increases from "negative infinity" near $x = 0$ through 0 at $x = 1$ and rises more and more slowly as x becomes larger (Figure 1.7.105). It is impossible to compute the natural log of a negative number (although more advanced fields of mathematics define these quantities using *complex numbers*).

Example 1.7.6 Exponential and Logarithmic Functions

- If $\ln(100) \approx 4.605$, then $e^{4.605} \approx 100$.
- If $e^5 \approx 148.41$, then $\ln(148.41) \approx 5$.
- If $\ln(0.1) \approx -2.303$, then $e^{-2.303} \approx 0.1$.
- If $e^{-3} \approx 0.04979$, then $\ln(0.04979) \approx -3$.

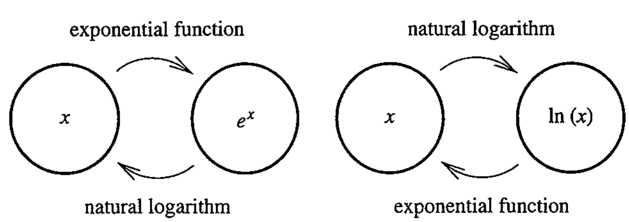


FIGURE 1.7.104
The exponential function and natural logarithm are inverses

The key to understanding natural logarithms is knowing the laws of logs, presented in the accompanying table, which are the laws of exponents in reverse.

The laws of logs (for x, $y > 0$, and any number p)

Law 1	$\ln(xy) = \ln(x) + \ln(y)$
Law 2	$\ln(x^p) = p\ln(x)$
Law 3	$\ln(1/x) = -\ln(x)$
Law 4	$\ln(x/y) = \ln(x) - \ln(y)$
Law 5	$\ln(e) = 1$
Law 6	$\ln(1) = 0$

Example 1.7.7 The Laws of Logs in Action

- $\ln(3) + \ln(4) = \ln(3 \cdot 4) = \ln(12)$, using law 1.
- $\ln(3) \cdot \ln(4)$ cannot be simplified with a law of logs.
- $\ln(3^4) = 4\ln(3)$, using law 2.
- $\ln(1/3) = -\ln(3)$, using law 3.
- $\ln(4/3) = \ln(4) - \ln(3)$, using law 4.

In some disciplines, people use the **exponential function with base 10,** or

$$f(x) = 10^x$$

Its inverse is the **logarithm to the base 10,** which is written

$$\log_{10} x$$

and is read "log base 10 of x." Just as $\ln(x) = y$ implies that $x = e^y$,

$$\log_{10} x = y$$

implies that

$$x = 10^y$$

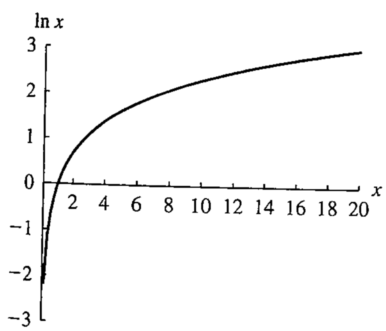


FIGURE 1.7.105
Graph of the natural logarithm

For example, if $\log_{10} x = 2.3$, then $x = 10^{2.3} \approx 199.5$. In most ways, the exponential function with base 10 and the log base 10 work much like the exponential function with base e and the natural logarithm. All laws of exponents and logs are the same except law 5, which becomes

Law 5 of exponents:	$10^1 = 10$
Law 5 of logs:	$\log_{10}(10) = 1$

The base e is more convenient for studying dynamics with calculus.

Example 1.7.8 Converting Logarithms in Base 10 to Natural Logs

Suppose $\log_{10}(x) = y$. How can we find $\ln(x)$? By the definition of $\log_{10}$,

$$x = 10^y$$

Then

$\ln(x) = \ln(10^y)$	take the natural log of both sides
$= y\ln(10)$	law 2 of logs
$\approx 2.303y$	because $\ln(10) \approx 2.303$

Rewriting in terms of $\log_{10}$, we find that

$$\ln(x) \approx 2.303 \log_{10}(x)$$

For instance, $\log_{10}(100) = 2$, so $\ln(100) \approx 2.303 \cdot 2 = 4.606$.

Expressing Results with Exponentials

We can use the laws of exponentials and logs to express

$$b_t = r^t b_0$$

in terms of the exponential function with base e. Because the exponential function and the natural logarithm are inverses, we can rewrite r as

$$r = e^{\ln(r)}$$

Then, using law 2 of exponents,

$$\begin{aligned} r^t &= \left(e^{\ln(r)}\right)^t \\ &= e^{\ln(r)t} \end{aligned}$$

Therefore, the **general solution** for the discrete-time dynamical system

$$b_{t+1} = rb_t$$

with initial condition b_0 can be written in exponential notation as

$$b_t = b_0 e^{\ln(r)t}$$

Example 1.7.9 Expressing a Solution with the Exponential Function

Consider the case $r = 2.0$ and $b_0 = 1.0$. Because $\ln(2.0) \approx 0.6931$, the solution is

$$b_t = 1.0e^{\ln(2.0)t} \approx 1.0e^{(0.6931)t}$$

What is the value of rewriting the solution in this way? Exponential notation makes it easier to answer questions about when a population will reach a particular value.

Example 1.7.10 Using a Solution Expressed with the Exponential Function: Increasing Case

When will the population described in the introduction, with solution

$$b_t = 2.0^t$$

reach 100.0? In Example 1.7.9 we wrote this solution in exponential notation. Now we can set $b_t = 100.0$ and solve for t with the steps

$$\begin{aligned} e^{\ln(2.0)t} &= 100.0 && \text{equation for } t \\ \ln(2.0)t &= \ln(100.0) && \text{take the natural log of both sides} \\ t &= \frac{\ln(100.0)}{\ln(2.0)} \approx 6.64 && \text{solve for } t \end{aligned}$$

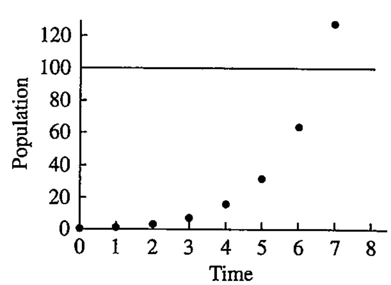


FIGURE 1.7.106
Using solutions to find times

The population will pass 100 million between hours 6 and 7 (Figure 1.7.106). The key step uses the natural log, the inverse of the exponential function, to remove the variable t from the exponent.

Example 1.7.11 Using a Solution Expressed with the Exponential Function: Decreasing Case

How long it will take a population with $r < 1$ to decrease to some specified value? Suppose $r = 0.7$ and $b_0 = 100$. The population decreases because $r < 1$. When will it reach $b_t = 2$? In exponential notation,

$$b_t = 100.0e^{\ln(0.7)t}$$

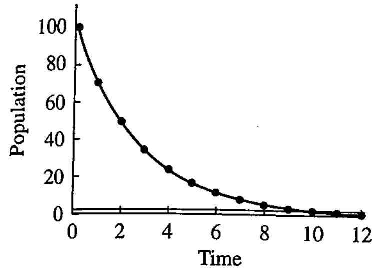


FIGURE 1.7.107
Using solutions to find times

Then $b_t = 2.0$ can be solved

$$100.0e^{\ln(0.7)t} = 2.0 \qquad \text{equation for } t$$
$$e^{\ln(0.7)t} = 0.02 \qquad \text{divide both sides by 100}$$
$$\ln(0.7)t = \ln(0.02) \qquad \text{take the natural log}$$
$$t = \frac{\ln(0.02)}{\ln(0.7)} \approx 10.97 \qquad \text{solve for } t$$

This population will pass 2.0 just before hour 11 (Figure 1.7.107).

Throughout the sciences, many measurements other than population sizes are described by exponential functions. In such cases, we write the measurement S as a function of t as

$$S(t) = S(0)e^{\alpha t}$$

The parameter $S(0)$ represents the value of the measurement at time $t = 0$. The parameter α describes how the measurement changes; α has dimensions of $1/\text{time}$. When $\alpha > 0$, the function is increasing (Figure 1.7.108a and b). When $\alpha < 0$, the function is decreasing (Figure 1.7.108c and d). The function increases most quickly with large positive values of α, and it decreases most quickly with large negative values of α.

One important number describing such measurements is the **doubling time.** When $\alpha > 0$, the measurement is increasing. A convenient measure of the speed of increase is the time it takes the initial value to double.

Example 1.7.12 Computing a Doubling Time from Scratch

Suppose

$$S(t) = 150.0e^{1.2t}$$

with t measured in hours. This measurement starts at $S(0) = 150.0$ and doubles when

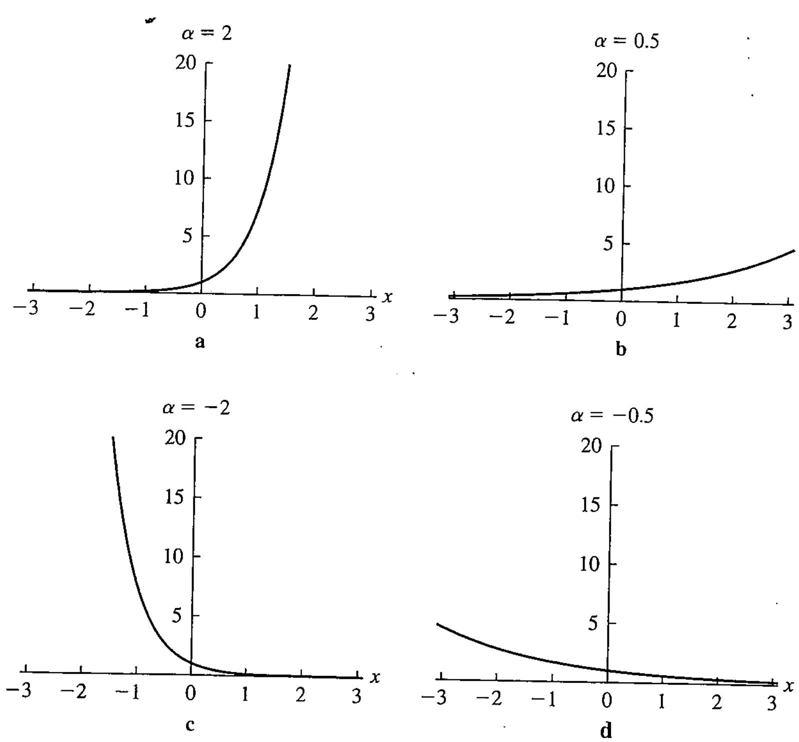


FIGURE 1.7.108
The exponential function with different parameter values in the exponent

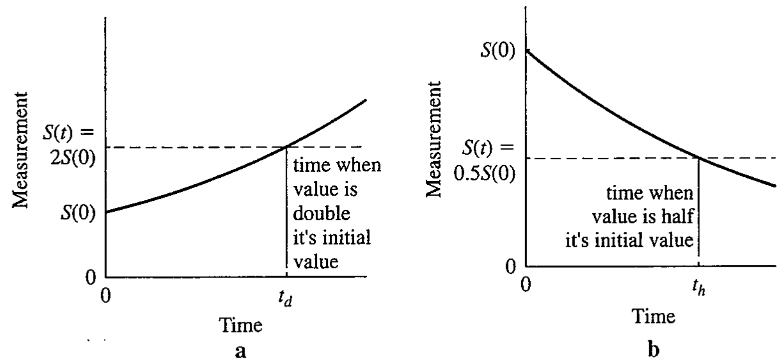


FIGURE 1.7.109
Doubling times and half-lives

$S(t) = 300.0$, or

$$\begin{aligned} 150.0e^{1.2t} &= 300.0 \\ e^{1.2t} &= 2.0 \\ 1.2t &= \ln(2.0) \\ t &= \frac{\ln(2.0)}{1.2} \approx 0.5776 \end{aligned}$$

As a check, we compute

$$S(0.5776) = 150.0e^{1.2 \cdot 0.5776} \approx 300.0$$

We can solve for the doubling time for a measurement following $S(t) = S(0)e^{\alpha t}$ by finding the time t_d when $S(t_d) = 2S(0)$,

$$\begin{aligned} S(t_d) = S(0)e^{\alpha t_d} &= 2S(0) && \text{equation for } t_d \\ e^{\alpha t_d} &= 2 && \text{divide by } S(0) \\ \alpha t_d &= \ln(2) && \text{take the natural log} \\ t_d &= \frac{\ln(2)}{\alpha} \approx \frac{0.6931}{\alpha} && \text{solve for } t_d \end{aligned}$$

The **general formula for the doubling time** is

$$t_d \approx \frac{0.6931}{\alpha}$$

The doubling time becomes smaller as α becomes larger, consistent with the fact that measurements with larger values of α increase more quickly.

Example 1.7.13 Computing a Doubling Time with the Formula

Suppose $S(t) = 150.0e^{1.2t}$ as in Example 1.7.13. Then $\alpha = 1.2$/hour, and the doubling time is

$$t_d \approx \frac{0.6931}{1.2} \approx 0.5776 \text{ hour}$$

When $\alpha < 0$, the measurement is decreasing, and we can ask how long it will take to become half as large. This time, denoted t_h, is called the **half-life** and can be found with the following steps.

$$\begin{aligned} S(t_h) = S(0)e^{\alpha t_h} &= 0.5S(0) && \text{equation for } t_h \\ e^{\alpha t_h} &= 0.5 && \text{divide by } S(0) \\ \alpha t_h &= \ln(0.5) && \text{take the natural log} \\ t_h &= \frac{\ln(0.5)}{\alpha} \approx -\frac{0.6931}{\alpha} && \text{solve for } t_h \end{aligned}$$

Therefore, the **general formula for the half-life** is

$$t_h \approx -\frac{0.6931}{\alpha}$$

The half-life becomes smaller when α grows larger in absolute value. Remember to apply this equation only when $\alpha < 0$.

Example 1.7.14 Computing the Half-Life

If a measurement follows the equation

$$M(t) = 240.0e^{-2.3t}$$

with t measured in seconds, then $\alpha = -2.3$/second and the half-life is

$$t_h \approx \frac{-0.6931}{-2.3} \approx 0.3013 \text{ second}$$

Example 1.7.15 Thinking in Half-Lives

Consider the measurement $M(t)$ given in Example 1.7.14, with a half-life of 0.3014 second. To figure out how much the value will have decreased in 2.0 seconds, we could plug into the original formula, finding

$$M(2.0) = 240.0e^{-2.3 \cdot 2.0} \approx 2.41$$

The value decreased by a factor of nearly 100. Alternatively, 2.0 seconds is

$$\frac{2.0}{0.3013} \approx 6.636$$

half-lives. After this many half-lives, the value will have decreased by a factor of $2^{6.636} \approx 99.46$, so that $M(2.0) \approx \frac{240.0}{99.46} \cong 2.41$. We can think of using half-lives as converting the exponential to base 2.

Conversely, if we are told the initial value and the doubling time or half-life of some measurement, we can find the formula. Instead of solving for the doubling time, we solve for the parameter α.

Example 1.7.16 Finding the Formula from the Doubling Time

Suppose $t_d = 26{,}200$ years for some measurement m. Because

$$t_d \approx \frac{0.6931}{\alpha}$$

we can solve for α as

$$\alpha \approx \frac{0.6931}{t_h} = \frac{0.6931}{26{,}200} = 2.645 \times 10^{-5}$$

If $m(0) = 0.031$, then the formula for $m(t)$ is

$$m(t) = 0.031e^{2.645 \times 10^{-5}t}$$

Example 1.7.17 Finding the Formula from the Half-Life

Suppose $t_h = 6.8$ years for some measurement V. Because

$$t_h \approx -\frac{0.6931}{\alpha}$$

we can solve for α as

$$\alpha \approx -\frac{0.6931}{t_h} = -\frac{0.6931}{6.8} \approx -0.1019$$

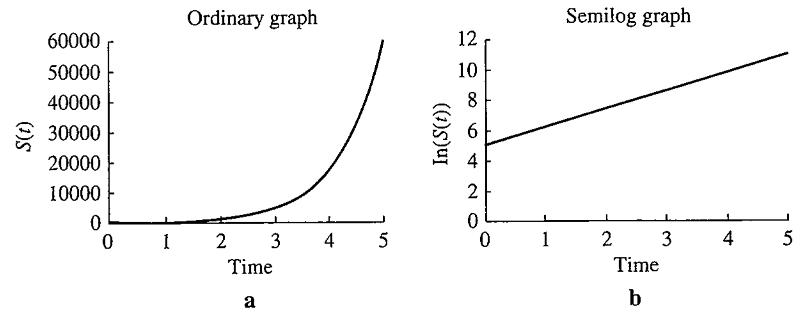


FIGURE 1.7.110
Original graph and semilog graph

If $V(0) = 23.1$, then the formula for $V(t)$ is

$$V(t) = 23.1e^{-0.1019t}$$

When a measurement follows an exponential function, the results are often plotted on a **semilog graph.**

Definition 1.13 A semilog graph plots the logarithm of the output against the input.

Example 1.7.18 A Semilog Graph of a Growing Value

Suppose

$$S(t) = 150.0e^{1.2t}$$

with t measured in hours (Example 1.7.13 and Figure 1.7.110a). To plot a semilog graph of $S(t)$ against t, we find the natural logarithm of $S(t)$.

$$\begin{aligned} \ln(S(t)) &= \ln\left(150.0e^{1.2t}\right) && \text{the natural logarithm of } S(t) \\ &= \ln(150.0) + \ln\left(e^{1.2t}\right) && \text{break up with law 2 of logs} \\ &\approx 5.01 + 1.2t && \text{evaluate } \ln(150.0) \text{ and cancel ln and exponent} \end{aligned}$$

Therefore, the semilog graph is a line with intercept 5.01 and slope 1.2 (Figure 1.7.110b) and transforms a curve into a line.

Example 1.7.19 A Semilog Graph of Some Data

Suppose we are to graph the following data.

Time	Value
0	120.12
1	24.34
2	2.19
3	0.89
4	0.056
5	0.078
6	0.125
7	0.346
8	1.128

The graph of the original data is difficult to read because the large vertical scale makes the small values almost indistinguishable (Figure 1.7.111a). If we take the logarithm of the data, however, the values are much easier to compare (Figure 1.7.111b).

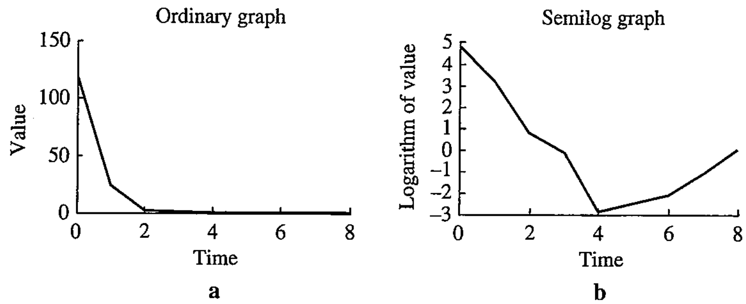


FIGURE 1.7.111
Original graph and semilog graph

Time	Value	Logarithm of Value
0	120.12	4.79
1	24.34	3.19
2	2.19	0.78
3	0.89	−0.12
4	0.056	−2.88
5	0.078	−2.55
6	0.125	−2.08
7	0.346	−1.06
8	1.128	0.12

We can see that the value reached a minimum at time 4 and increased steadily after that.

Summary We generalized the discrete-time dynamical system for bacterial population growth to compute when populations would reach particular values. If some offspring die, the discrete-time dynamical system can be written in terms of the **per capita production** r. A population grows if $r > 1$ and declines if $r < 1$. The solution can be expressed as an **exponential function to the base** r. For convenience, exponential functions are often expressed to the base e, often called the **exponential function.** Using the laws of exponents, any exponential function can be expressed to the base e. The inverse of the exponential function is the **natural logarithm** or natural log. This function can be used to solve equations involving the exponential function, including finding **doubling times** and **half-lives.** Measurements that cover a large range of positive values can be conveniently displayed on a **semilog graph,** which reduces the range and which produces a linear graph if the measurements follow an exponential function.

1.7 Exercises

Mathematical Techniques

1–10 ▪ Use the laws of exponents to rewrite the following, if possible. If no law of exponents applies, say so.

1. 43.2^0

2. 43.2^1

3. 43.2^{-1}

4. $43.2^{-0.5} + 43.2^{0.5}$

5. $43.2^{7.2}/43.2^{6.2}$

6. $43.2^{0.23} \cdot 43.2^{0.77}$

7. $(3^4)^{0.5}$

8. $(43.2^{-1/8})^{16}$

9. $2^{2^3} \cdot 2^{2^2}$

10. $4^2 \cdot 2^4$

11–20 ▪ Use the laws of logs to rewrite the following, if possible. If no law of logs applies or the quantity is not defined, say so.

11. $\ln(1)$

12. $\ln(-6.5)$

13. $\log_{43.2} 43.2$

14. $\log_{10}(3.5 + 6.5)$

15. $\log_{10}(5) + \log_{10}(20)$
16. $\log_{10}(0.5) + \log_{10}(0.2)$
17. $\log_{10}(500) - \log_{10}(50)$
18. $\log_{43.2}(5 \cdot 43.2^2) - \log_{43.2}(5)$
19. $\log_{43.2}(43.2^7)$
20. $\log_{43.2}(43.2^7)^4$

21–22 ▪ Apply the laws of logs with base equal to 7 to compute the following.

21. Using the fact that $\log_7 43.2 \approx 1.935$, find $\log_7\left(\frac{1}{43.2}\right)$.
22. Using the fact that $\log_7 43.2 \approx 1.935$, find $\log_7 [(43.2)^3]$.

23–26 ▪ Solve the following equations for x. Plug in your answer to check.

23. $7e^{3x} = 21$
24. $4e^{2x+1} = 20$
25. $4e^{-2x+1} = 7e^{3x}$
26. $4e^{2x+3} = 7e^{3x-2}$

27–30 ▪ Sketch graphs of the following exponential functions. For each, find the value of x where the function is equal to 7.0. For the increasing functions, find the doubling time, and for the decreasing functions, find the half-life. For what value of x is the value of the function 3.5? For what value of x is the value of the function 14.0?

27. e^{2x}
28. e^{-3x}
29. $5e^{0.2x}$
30. $0.1e^{-0.2x}$

31–32 ▪ Sketch graphs of the following updating functions over the given range, and mark the equilibria.

31. $h(z) = e^{-z}$ for $0 \le z \le 2$
32. $F(x) = \ln(x) + 1$ for $0 \le x \le 2$. (Although the equilibria cannot be found algebraically, you can guess the answer.)

Applications

33–36 ▪ Find the solution of each discrete-time dynamical system, express it in exponential notation, and solve for the time when the value reaches the given target. Sketch a graph of the solution.

33. A population follows the discrete-time dynamical system $b_{t+1} = rb_t$ with $r = 1.5$ and $b_0 = 1.0 \times 10^6$. When will the population reach 1.0×10^7?
34. A population follows the discrete-time dynamical system $b_{t+1} = rb_t$ with $r = 0.7$ and $b_0 = 5.0 \times 10^5$. When will the population reach 1.0×10^5?
35. Cell volume follows the discrete-time dynamical system $v_{t+1} = 1.5v_t$ with initial volume of 1350 μm^3 (as in Section 1.5, Exercise 37). When will the volume reach 3250 μm^3?
36. Gnat number follows the discrete-time dynamical system $n_{t+1} = 0.5n_t$ with an initial population of 5.5×10^4. When will the population reach 1.5×10^3?

37–40 ▪ Suppose the size of an organism at time t is given by

$$S(t) = S_0 e^{\alpha t}$$

where S_0 is the initial size. Find the time it takes for the organism to double and to quadruple in size in the following circumstances.

37. $S_0 = 1.0$ cm and $\alpha = 1.0$/day
38. $S_0 = 2.0$ cm and $\alpha = 1.0$/day
39. $S_0 = 2.0$ cm and $\alpha = 0.1$/hour
40. $S_0 = 2.0$ cm and $\alpha = 0.0$/hour

41–42 ▪ Suppose the size of an organism at time t is given by

$$S(t) = S_0 10^{\alpha t}$$

where S_0 is the initial size and t is measured in days. Find the time it takes for the organism to double in size by converting to base e. How long will it take to increase by a factor of 10?

41. $S_0 = 2.34$ and $\alpha = 0.5$
42. $S_0 = 2.34$ and $\alpha = 0.693$

43–46 ▪ The amount of carbon-14 (C^{14}) left t years after the death of an organism is given by

$$Q(t) = Q_0 e^{-0.000122t}$$

where Q_0 is the amount left at the time of death. Suppose $Q_0 = 6.0 \times 10^{10}$ C^{14} atoms.

43. How much is left after 50,000 years? What fraction is this of the original amount?
44. How much is left after 100,000 years? What fraction is this of the original amount?
45. Find the half-life of C^{14}.
46. About how many half-lives will occur in 50,000 years? Roughly what fraction will be left? How does this compare with the answer to Exercise 43?

47–50 ▪ Suppose a population has a doubling time of 24 years and an initial size of 500.

47. What is the population in 48 years?
48. What is the population in 12 years?
49. Find the equation for population size $P(t)$ as a function of time.
50. Find the 1-year discrete-time dynamical system for this population (figure out the factor multiplying the population in 1 year).

51–54 ▪ Suppose a population is dying with a half-life of 43 years. The initial size is 1600.

51. How long will it take to reach 200?
52. Find the population in 86 years.
53. Find the equation for population size $P(t)$ as a function of time.
54. Find the 1-year discrete-time dynamical system for this population (figure out the factor multiplying the population in 1 year).

55–58 ▪ Plot semilog graphs of the values from the earlier problems.

55. The growing organism in Exercise 37 for $0 \le t \le 10$. Mark where the organism has doubled in size and where it has quadrupled in size.

56. The carbon-14 in Exercise 43 for $0 \le t \le 20000$. Mark where the amount of carbon has gone down by half.

57. The population in Exercise 47 for $0 \le t \le 100$. Mark where the population has doubled.

58. The population in Exercise 51 for $0 \le t \le 100$. Mark where the population has gone down by half.

Computer Exercises

59. Use your computer to find the following. Plot the graphs to check.

a. The doubling time of $S_1(t) = 3.4e^{0.2t}$.

b. The doubling time of $S_2(t) = 0.2e^{3.4t}$.

c. The half-life of $H_1(t) = 3.4e^{-0.2t}$.

d. The half-life of $H_2(t) = 0.2e^{-3.4t}$.

60. Have your computer solve for the times when the following hold. Plot the graphs to check your answers.

a. $S_1(t) = S_2(t)$ with S_1 and S_2 from the previous problem.

b. $H_1(t) = 2H_2(t)$ with H_1 and H_2 from the previous problem.

c. $H_1(t) = 0.5H_2(t)$ with H_1 and H_2 from the previous problem.

61. Use your computer to plot the following functions.

a. $\ln(x)$ for $10 \le x \le 100{,}000$

b. $\ln(\ln(x))$ for $10 \le x \le 100{,}000$

c. $\ln(\ln(\ln(x)))$ for $10 \le x \le 100{,}000$

d. e^x for $0 \le x \le 2$

e. e^{e^x} for $0 \le x \le 2$

f. $e^{e^{e^x}}$ for $0 \le x \le 2$. Will your machine let you do it? Can you compute the value of $e^{e^{e^2}}$?

62. Use your computer to compute the following. Does this give you any idea why e is special?

a. $2^{0.001}$

b. $10^{0.001}$

c. $0.5^{0.001}$

d. $e^{0.001}$

1.8 Oscillations and Trigonometry

We have used linear and exponential functions to describe several types of relations between measurements. Important as they are, these functions cannot describe **oscillations,** processes that repeat in cycles. Heartbeats and breathing are examples of biological oscillations. In addition, the daily and seasonal cycles imposed by the movements of the earth drive sleep-wake cycles, seasonal population cycles, and the tides. In this section, we will use **trigonometric functions** to describe simple oscillations. Four numbers are needed to describe such oscillations with the **cosine** function: the **average,** the **amplitude,** the **period,** and the **phase.**

Sine and Cosine: A Review

Like many functions, the trigonometric functions have two interpretations: geometric and dynamical. Geometrically, the trigonometric functions are used to compute angles and distances. After briefly reviewing the geometry behind the **sine** and **cosine** functions, we will use them to study the dynamics of biological oscillations.

In applied mathematics, angles are measured in **radians.** For an angle with vertex at the center of a circle of radius 1, its measure in radians is equal to the length of the arc of the circle subtended (lying inside) the angle (Figure 1.8.112). Because the full circumference of a circle with radius 1 is 2π, 2π radians corresponds to 360°, or one complete revolution. There is thus a **basic identity** between radians and degrees given by

$$2\pi \text{ radians} = 360°$$

From this, we derive the **conversion factors**

$$1 = \frac{2\pi \text{ radians}}{360°} = \frac{\pi \text{ radians}}{180°}$$

$$1 = \frac{360°}{2\pi \text{ radians}} = \frac{180°}{\pi \text{ radians}}$$

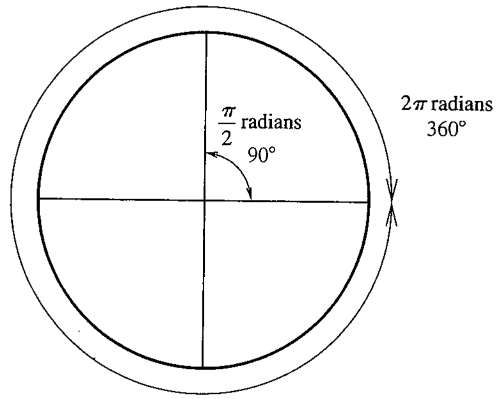


FIGURE 1.8.112
Degrees and radians

Example 1.8.1 Converting Degrees to Radians

To find 60° in radians, we convert

$$60° = 60° \times \frac{\pi \text{ radians}}{180°} = \frac{\pi}{3}\text{radians}$$

Example 1.8.2 Converting Radians to Degrees

Similarly, to find 1.0 radians in degrees, we convert

$$1.0\,\text{radians} = 1.0\,\text{radians} \times \frac{180°}{\pi \text{ radians}} \approx 57.3°$$

The **sine function** and the **cosine function** take angles as inputs and return numbers between -1 and 1 as outputs. We write $\sin(\theta)$ and $\cos(\theta)$ to denote these functions, where the Greek letter θ (theta) is often used for angles. The sine and cosine give the Cartesian coordinates of points on the circle (Figure 1.8.113).

Definition 1.14 The Cartesian coordinates of the point on the unit circle an angle θ measured counterclockwise from $(1, 0)$ are $(\cos(\theta), \sin(\theta))$.

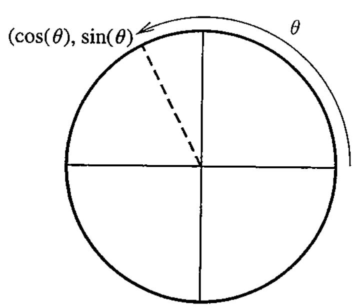


FIGURE 1.8.113
The definition of $\sin(\theta)$ and $\cos(\theta)$

Values of these functions for representative inputs are given in the following table.

Radians	Degrees	$\cos(\theta)$	$\sin(\theta)$	Radians	Degrees	$\cos(\theta)$	$\sin(\theta)$
0	0°	1	0	π	180°	-1	0
$\frac{\pi}{6}$	30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{7\pi}{6}$	210°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{5\pi}{4}$	225°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{4\pi}{3}$	240°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	90°	0	1	$\frac{3\pi}{2}$	270°	0	-1
$\frac{2\pi}{3}$	120°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{5\pi}{3}$	300°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	135°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{7\pi}{4}$	315°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	150°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{11\pi}{6}$	330°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
π	180°	-1	0	2π	360°	1	0

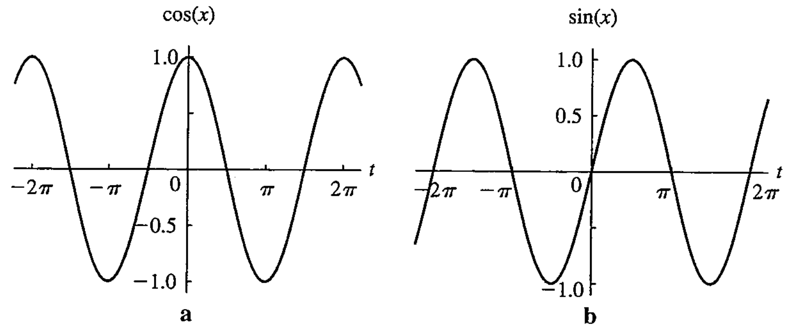


FIGURE 1.8.114
Graphs of the cosine and sine functions

Both the sine and cosine functions repeat every 2π radians (Figure 1.8.114). The value 2π is called the **period** of the oscillation. This means that adding (or subtracting) multiples of 2π to (or from) the argument does not change the value, so

$$\cos(\theta) = \cos(\theta + 2\pi) = \cos(\theta + 4\pi) = \cos(\theta + 2n\pi)$$
$$\cos(\theta) = \cos(\theta - 2\pi) = \cos(\theta - 4\pi) = \cos(\theta - 2n\pi)$$

for any value of θ and any integer n and similarly for the sine function.

Example 1.8.3 Periodicity of the Cosine Function

$$\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} + 2\pi\right) = \cos\left(\frac{\pi}{4} + 4\pi\right) = \frac{\sqrt{2}}{2}$$
$$\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} - 2\pi\right) = \cos\left(\frac{\pi}{4} - 4\pi\right) = \frac{\sqrt{2}}{2}$$

The graphs of sine and cosine have the same shape but are **shifted** from each other by $\pi/2$ rad (Figure 1.8.114). In equations,

$$\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$$

Example 1.8.4 Relation Between Sine and Cosine

$$\sin\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Because we can compute the sine function in terms of the cosine function, we will use cosine to describe oscillations.

Describing Oscillations with the Cosine

A measurement is said to **oscillate** as a function of time if the values vary regularly between high and low values. Oscillations that are shaped like the graph of the sine or cosine function are called **sinusoidal.** There are four numbers needed to describe an oscillation with the cosine function: the **average,** the **amplitude,** the **period,** and the **phase** (Figure 1.8.115).

- The **amplitude** is the difference between the maximum and the average (or the average and the minimum).
- The **average** lies halfway between the minimum and maximum values.
- The **period** is the time between successive peaks.
- The **phase** is the time of the first peak.

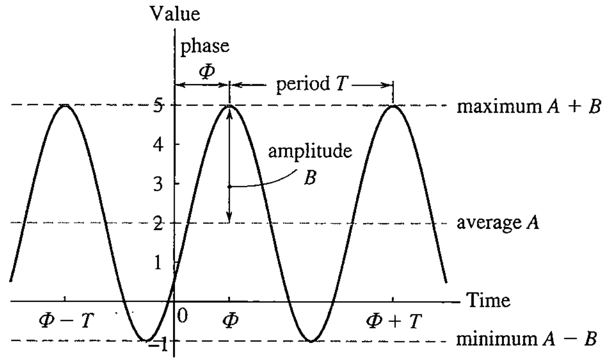


FIGURE 1.8.115
The four numbers that describe a sinusoidal oscillation

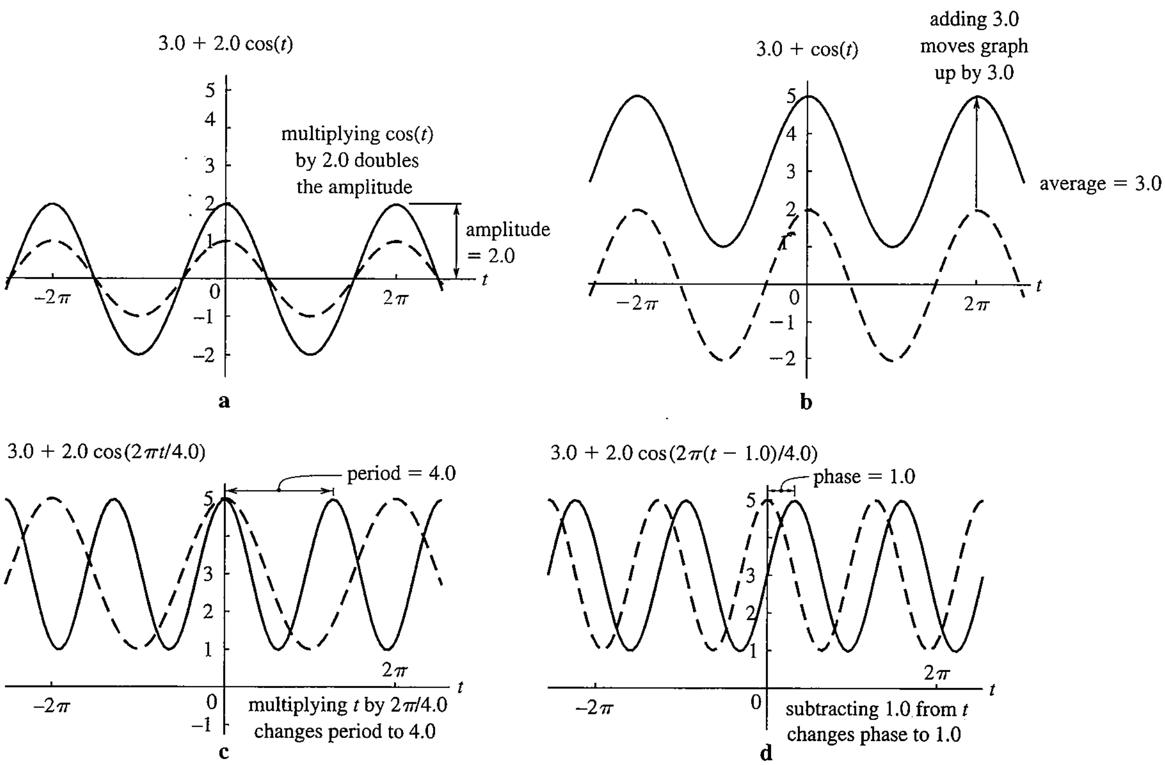


FIGURE 1.8.116
Building a function with different average, amplitude, period, and phase

We can build the oscillation shown in Figure 1.8.115 from the cosine function by **shifting** and **scaling** both vertically and horizontally (Section 1.3).

Example 1.8.5 Building an Oscillation by Shifting and Scaling the Cosine Function

Suppose we wish to build a function with an amplitude of 2.0, an average of 3.0, a period of 4.0, and a phase of 1.0. We can construct the formula in steps.

1. To increase the amplitude by a factor of 2.0, we scale vertically by **multiplying** the cosine by 2.0 (Figure 1.8.116a). The function is now

$$f(t) = 2.0\cos(t)$$

2. To raise the **average** from 0 to 3.0, we vertically shift the function by **adding** 3.0 to the function (Figure 1.8.116b), making

$$f(t) = 3.0 + 2.0\cos(t)$$

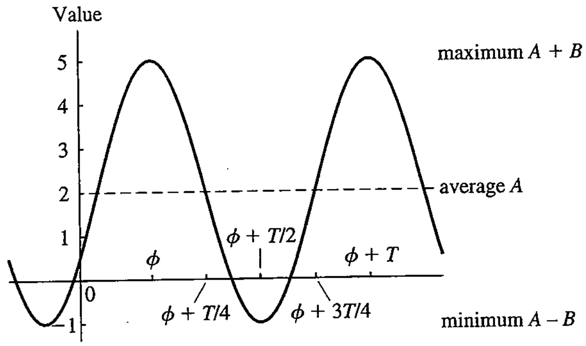


FIGURE 1.8.117
The guideposts for plotting $f(t) = A + B\cos\left(\frac{2\pi}{T}(t-\phi)\right)$

3. Next, we wish to decrease the period from 2π to 4.0. We do this by scaling horizontally by a factor of $\frac{2\pi}{4}$, or by **multiplying** the t inside the cosine by $\frac{2\pi}{4.0}$ (Figure 1.8.116c). Our function is now
$$f(t) = 3.0 + 2.0\cos\left(\frac{2\pi}{4.0}t\right)$$

4. Finally, we shift the curve horizontally so that the first peak is at 1.0 instead of 0.0. We do this by **subtracting** 1.0 from t (Figure 1.8.116d), arriving at the final answer of
$$f(t) = 3.0 + 2.0\cos\left(\frac{2\pi}{4.0}(t-1.0)\right)$$

In general, a sinusoidal oscillation with amplitude B, average A, period T, and phase ϕ (phi) can be described as a function of time t with the formula
$$f(t) = A + B\cos\left(\frac{2\pi}{T}(t-\phi)\right) \tag{1.8.1}$$

This function has a maximum at $t = \phi$, has a minimum at $t = \phi + \frac{T}{2}$, and takes on its average value at $t = \phi + \frac{T}{4}$ and $t = \phi + \frac{3T}{4}$. Thereafter, it repeats every T (Figure 1.8.117).

Example 1.8.6 Plotting a Sinusoidal Function from its Equation

Suppose we wish to plot
$$f(t) = 2.0 + 0.4\cos\left(\frac{2\pi}{10.0}(t-7.0)\right)$$

The amplitude is 0.4, and the average is 2.0, the period is 10.0, and the phase is 7.0 (Figure 1.8.118). The maximum is the sum of the average and amplitude, or $2.0 + 0.4 = 2.4$, and the minimum is the average minus the amplitude, or $2.0 - 0.4 = 1.6$. The first

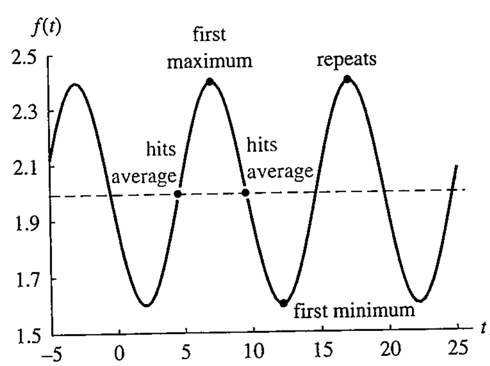


FIGURE 1.8.118
Graphing a sinusoidal oscillation from its equation

maximum occurs at the phase, or at $t = 7.0$. The average occurs 1/4 and 3/4 of the way through each cycle, or at $t = 7.0 + \frac{1}{4}(10.0) = 9.5$ and $t = 7.0 + \frac{3}{4}(10.0) = 14.5$. The minimum occurs halfway through the first period at $t = 7.0 + \frac{1}{2}(10.0) = 12.0$. The cycle repeats at $t = 17.0$, 27.0, and so forth.

Example 1.8.7 The Daily and Monthly Temperature Cycles

Women have two cycles affecting body temperature: a daily and a monthly rhythm. The key facts about these two cycles are given in the following table.

	Minimum	Maximum	Average	Time of Maximum	Period
Daily cycle	36.5	37.1	36.8	2:00 P.M.	24 hours
Monthly cycle	36.6	37.0	36.8	Day 16	28 days

Assuming that these cycles are sinusoidal, we can use this information to describe these cycles with the cosine function.

The amplitude of a cycle is

$$\text{amplitude} = \text{maximum} - \text{average}$$

For the daily cycle, the amplitude is

$$\text{daily cycle amplitude} = 37.1 - 36.8 = 0.3$$

For the monthly cycle, the amplitude is

$$\text{monthly cycle amplitude} = 37.0 - 36.8 = 0.2$$

The phase depends on the time chosen as the starting time. We define the daily cycle to begin at midnight and the monthly cycle to begin at menstruation. The maximum of the daily cycle occurs 14 hours after the start, and that of the monthly cycle 16 days after the start. The oscillations can be described by the fundamental formula (Equation 1.8.1). For the daily cycle, with t measured in hours, the formula $P_d(t)$ is

$$P_d(t) = 36.8 + 0.3\cos\left(\frac{2\pi(t-14)}{24}\right)$$

(Figure 1.8.119a.) For the monthly cycle, with t measured in days, the formula $P_m(t)$ is

$$P_m(t) = 36.8 + 0.2\cos\left(\frac{2\pi(t-16)}{28}\right)$$

(Figure 1.8.119b.)

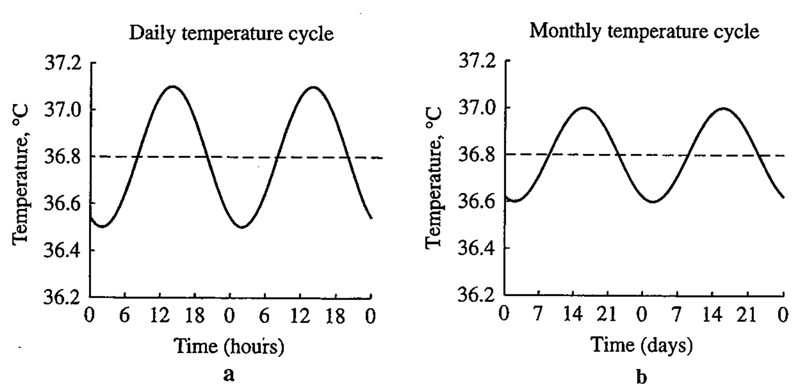


FIGURE 1.8.119
The daily and monthly temperature cycles

More Complicated Shapes

Real oscillations are not perfectly sinusoidal. Nonetheless, the cosine function is useful for describing more complicated oscillations. A powerful theory beyond the scope of this book, called **Fourier series,** shows how almost any oscillation can be written as the sum of many cosine functions with different amplitudes, periods, and phases (see Exercise 55).

As an illustration, we will combine the daily and monthly temperature cycles. To do so, we must write both cycles in the same time units, days. In days, the period of the daily cycle is 1.0 day and the phase (the time of the maximum) is

$$\text{phase in days} = 14 \text{ hours} \cdot \frac{1.0 \text{ days}}{24 \text{ hours}} \approx 0.583 \text{ day}$$

The equation for the daily cycle, with t measured in days, is

$$P_d(t) = 36.8 + 0.3\cos(2\pi(t - 0.583))$$

To figure out how the daily and monthly cycles combine, we cannot simply add them together, because

$$\begin{aligned} P_d(t) + P_m(t) &= 36.8 + 0.3\cos(2\pi(t - 0.583)) + 36.8 + 0.2\cos\left(\frac{2\pi(t-16)}{28}\right) \\ &= 73.6 + 0.3\cos(2\pi(t - 0.583)) + 0.2\cos\left(\frac{2\pi(t-16)}{28}\right) \end{aligned}$$

which has an average of 73.6.

To keep the average at the appropriate value of 36.8, we add only the two cosine terms to the average, getting a formula for the combined cycle of

$$P_t(t) = 36.8 + 0.2\cos\left(\frac{2\pi(t-16)}{28}\right) + 0.3\cos(2\pi(t - 0.583))$$

(Figure 1.8.120). In the course of one month, there is a single slow cycle, with 28 daily cycles superimposed. The maximum possible temperature can be found by adding the sum of the amplitudes of the daily and monthly cycles to the overall average. That is,

$$\text{maximum possible temperature} = 36.8 + (0.2 + 0.3) = 37.3$$

This maximum occurs only if each cycle takes on its maximum at the same time, which does not happen exactly in this case. It is closest at 2:00 P.M. on the 16th day of the cycle.

The minimum possible temperature can be found by subtracting the sum of the amplitudes of the daily and monthly cycles from the overall average, or

$$\text{minimum possible temperature} = 36.8 - (0.2 + 0.3) = 36.3$$

This minimum occurs only if each cycle takes on its minimum at the same time, which also does not happen exactly. It is closest at 2:00 A.M. on the 2nd day of the cycle.

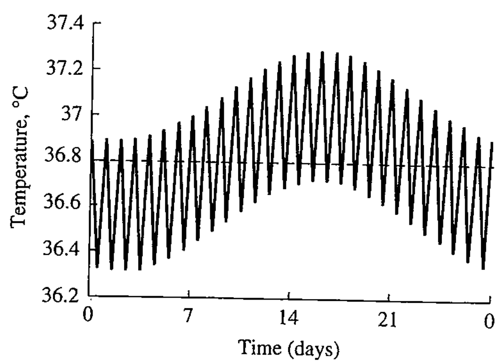


FIGURE 1.8.120
The combined effect of the daily and monthly temperature cycles

Summary **Sinusoidal** oscillations can be described mathematically with the **cosine** function. Four factors change the shape of the graph: the **amplitude** (the distance from the middle to the minimum or maximum), the **average** (the middle value), the **period** (the time between successive maxima), and the **phase** (the time of the first maximum). Functions with these parameters can be created by **shifting** and **scaling** the cosine function vertically and horizontally. Oscillations with more complicated shapes can be described by adding together appropriate cosine functions.

1.8 Exercises

Mathematical Techniques

1–6 ▪ Use a table or a calculator to find the values of sine and cosine for the following inputs (all in radians), and plot them **a.** on a graph of $\sin(\theta)$, **b.** on a graph of $\cos(\theta)$, **c.** as the coordinates of a point on the circle.

1. $\theta = \pi/2$

2. $\theta = 3\pi/4$

3. $\theta = \pi/9$

4. $\theta = 5.0$

5. $\theta = -2.0$

6. $\theta = 3.2$

7–14 ▪ Convert the following angles from degrees to radians, or vice versa.

7. 30°

8. 330°

9. 1°

10. −30°

11. 2.0 radians

12. $\pi/5$ radians

13. $-\pi/5$ radians

14. 30 radians

15–20 ▪ There are 360° in a full circle because the ancient Babylonians were fond of the number 60 and its multiples (as an approximation to the 365 days in a year). In the 1790s, the French introduced another system for measuring angles called "grads," where 400 grads make up a full circle (or 100 grads make up 90°). Although they are found on many calculators, these units are almost never used. Write the basic identities between degrees and grads, and between radians and grads, and use them to make the following conversions.

15. 180° into grads

16. 60° into grads

17. $\pi/4$ radians into grads

18. 3.0 radians into grads

19. 150 grads into degrees

20. 250 grads into radians

21–26 ▪ The other trigonometric functions (tangent, cotangent, secant, and cosecant) are defined in terms of sin and cos by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}$$
$$\sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}$$

Calculate the value of each of these functions at the following angles (all in radians). Plot the points on a graph of each function.

21. $\pi/2$

22. $3\pi/4$

23. $\pi/9$

24. 5.0

25. −2.0

26. 3.2

27–32 ▪ The following are some of the most important trigonometric identities. Check each of them at the points **a.** $\theta = 0$, **b.** $\theta = \pi/4$, **c.** $\theta = \pi/2$, **d.** $\theta = \pi$.

27. $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos(\theta)}{2}}$ for $0 \le \theta \le \pi$ (using the positive square root). Check only at points **a**, **c**, and **d**.

28. $\sin^2(\theta) + \cos^2(\theta) = 1$

29. $\cos(\theta - \pi) = -\cos(\theta)$

30. $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$

31. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

32. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

33–36 ▪ The following are alternative ways to write formulas for sinusoidal oscillations. Convert them to the standard form (Equation 1.8.1) and sketch a graph.

33. $r(t) = 5.0(2.0 + 1.0\cos(2\pi t))$

34. $g(t) = 2.0 + 1.0\sin(t)$. Use Exercise 30 to change the sine into cosine.

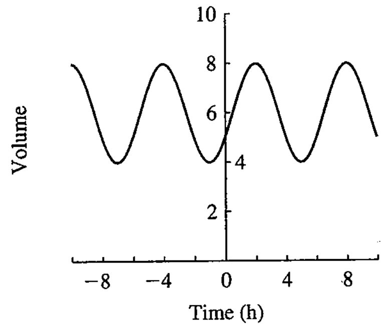
35. $f(t) = 2.0 - 1.0\cos(t)$. Use Exercise 29 to get rid of the negative amplitude.

36. $h(t) = 2.0 + 1.0\cos(2\pi t - 3.0)$ (the factor 2π does not multiply the 3.0).

Applications

37–40 ▪ Find the average, minimum, maximum, amplitude, period, and phase from the graphs of the following oscillations.

37.



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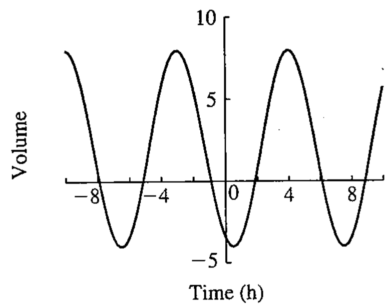

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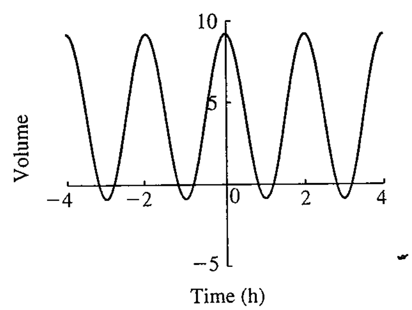

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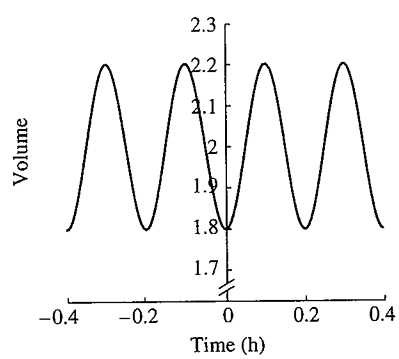


41–44 ▪ Graph the following functions. Give the average, maximum, minimum, amplitude, period, and phase of each and mark them on your graph.

41. $f(x) = 3.0 + 4.0\cos\left(2\pi\frac{x-1.0}{5.0}\right)$

42. $g(t) = 4.0 + 3.0\cos(2\pi(t-5.0))$

43. $h(z) = 1.0 + 5.0\cos\left(2\pi\frac{z-3.0}{4.0}\right)$

44. $W(y) = -2.0 + 3.0\cos\left(2\pi\frac{y+0.1}{0.2}\right)$

45–50 ▪ Oscillations are often combined with growth or decay. Plot graphs of the following functions, and describe in words what you see. Make up a biological process that might have produced the result.

45. $f(t) = 1 + t + \cos(2\pi t)$ for $0 < t < 4$, where t is measured in days.

46. $h(t) = t + 0.2\sin(2\pi t)$ for $0 < t < 4$, where t is measured in days.

47. $g(t) = e^t\cos(2\pi t)$ for $0 < t < 3$, where t is measured in years.

48. $W(t) = e^{-t}\cos(2\pi t)$ for $0 < t < 3$, where t is measured in years.

49. $H(t) = \cos(e^t)$ for $0 < t < 3$, where t is measured in years.

50. $b(t) = \cos(e^{-t})$ for $0 < t < 3$, where t is measured in years.

51–54 ▪ Sleepiness has two cycles, a circadian rhythm with a period of approximately 24 hours and an ultradian rhythm with a period of approximately 4 hours. Both have phase 0 and average 0, but the amplitude of the circadian rhythm is 1.0 sleepiness unit, and that of the ultradian is 0.4 sleepiness unit.

51. Find the formula and sketch the graph of sleepiness over the course of a day due to the circadian rhythm.

52. Find the formula and sketch the graph of sleepiness over the course of a day due to the ultradian rhythm.

53. Sketch the graph of the two cycles combined.

54. At what time of day are you sleepiest? At what time of day are you least sleepy?

Computer Exercises

55. Consider the following functions.

$$f_1(x) = \cos\left(x - \frac{\pi}{2}\right)$$

$$f_3(x) = \frac{\cos\left(3x - \frac{\pi}{2}\right)}{3}$$

$$f_5(x) = \frac{\cos\left(5x - \frac{\pi}{2}\right)}{5}$$

$$f_7(x) = \frac{\cos\left(7x - \frac{\pi}{2}\right)}{7}$$

a. Plot them all on one graph.

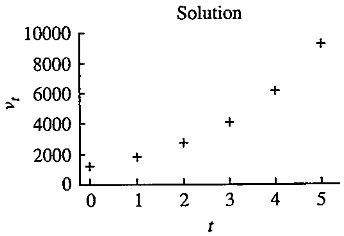
b. Plot the sum $f_1(x) + f_3(x)$.

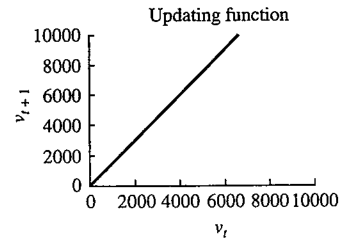
c. Plot the sum $f_1(x) + f_3(x) + f_5(x)$.

d. Plot the sum $f_1(x) + f_3(x) + f_5(x) + f_7(x)$.

15. $v_t = 1.5^t \cdot 1220\ \mu\text{m}^3$.

$$v_1 = 1.5 \cdot 1220 = 1830$$
$$v_2 = 1.5 \cdot 1830 = 2745$$
$$v_3 = 1.5 \cdot 2745 = 4117.5$$
$$v_4 = 1.5 \cdot 4117.5 = 6176.25$$
$$v_5 = 1.5 \cdot 6176.25 = 9264.375.$$





19. Plugging $t = 20$ into the solution $v_t = 1.5^t \cdot 1220\ \mu\text{m}^3$, we get $v_{20} = 4.05 \times 10^6\ \mu\text{m}^3$. This might be reasonable.

23. $x_1 = 1/2$, $x_2 = 1/3$, $x_3 = 1/4$, $x_4 = 1/5$. It looks like $x_t = \frac{1}{1+t}$.

25. $x_1 = 3$, $x_2 = 1$, $x_3 = 3$, $x_4 = 1$. It seems to be jumping back and forth between 1 and 3. If I start at $x_0 = 0$, the results jump back and forth between 0 and 4.

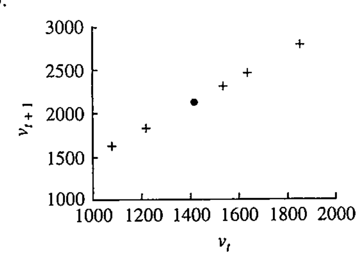
27. These do not commute. If you started with 100, doubled (giving 200), and then removed 10, you would end up with 190. If you started with 100, removed 10 (leaving 90), and then doubled, you'd have only 180. In general, if we call the starting population P_t, if we double first and then remove 10, we end up with $P_{t+1} = 2P_t - 10$. If we first remove 10 and then double, we end up with $P_{t+1} = 2(P_t - 10) = 2P_t - 20$, which never matches the result in the other order.

31. $h_{20} = 10.0 + 20 = 30.0$ m, a reasonable height for a tree.

33. 1.05×10^6 million bacteria. These will weigh about 10^{-6} g, which sounds reasonable.

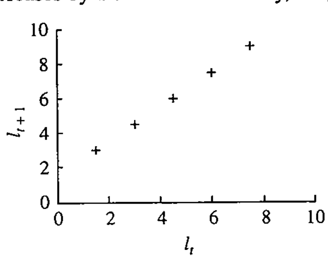
35. The solution is $x_1 = 50$, $x_2 = 130$, $x_3 = 290$. Adding 30, we see that $x_0 + 30 = 40$, $x_1 + 30 = 80 = 40 \cdot 2$, $x_2 + 30 = 160 = 40 \cdot 2^2$, and $x_3 + 30 = 320 = 40 \cdot 2^3$. It looks like $x_t + 30 = 40 \cdot 2^t$, so $x_t = 40 \cdot 2^t - 30$.

37.



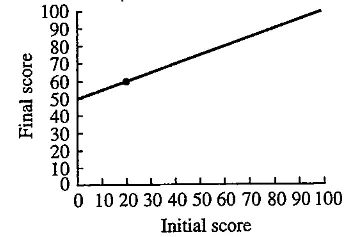
The discrete-time dynamical system is $v_{t+1} = 1.5 v_t$ and the missing value is 2130.

41. The length increases by 1.5 cm each half-day, so $l_{t+1} = l_t + 1.5$ cm.



45. The argument is the initial score. The value is the final score.

47.

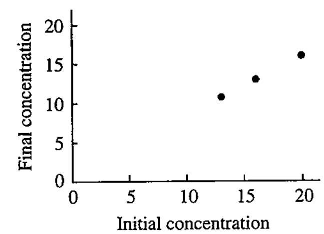


49. Let v_{t+1} and v_t be the total volume before and after the experiment. Then $v_t = 10^4 b_t$ and $v_{t+1} = 10^4 b_{t+1}$. The original discrete-time dynamical system is $b_{t+1} = 2.0 b_t$. Therefore, $v_{t+1} = 10^4 b_{t+1} = 10^4(2.0 b_t) = 2.0 \cdot 10^4 b_t = 2.0 v_t$.

51. $V_t = \pi h_t 0.5^2$, and $V_{t+1} = \pi h_{t+1} 0.5^2$. Therefore,

$$V_{t+1} = \pi(h_t + 1)0.5^2 = \pi h_t \cdot 0.5^2 + \pi \cdot 0.5^2 = V_t + \pi \cdot 0.5^2.$$

53. The points for the first patient are (20.0, 16.0), (16.0, 13.0), and (13.0, 10.75).



Let the level be M_t at the beginning of the day. The two points are (20.0, 16.0) and (16.0, 13.0). The slope is

$$\text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{13.0 - 16.0}{16.0 - 20.0} = 0.75.$$

In point-slope form, the discrete-time dynamical system is

$$M_{t+1} = 0.75\,(M_t - 20.0) + 16.0 = 0.75 M_t + 1.0.$$

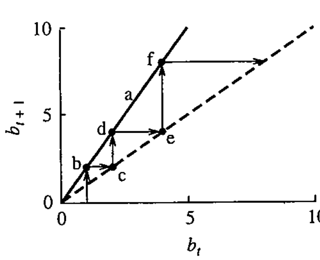
55. The solution for the first is $b_t = 2.0^t \cdot 1.0 \times 10^6$, and the solution for the second is $b_t = 2.0^t \cdot 3.0 \times 10^5$. The difference is $2.0^t \cdot 0.7 \times 10^6$, but the ratio is always approximately 3.33. Both populations are growing at the same rate, but the first has a head start. It is always 3.33 times bigger, which becomes a larger difference as the populations become larger.

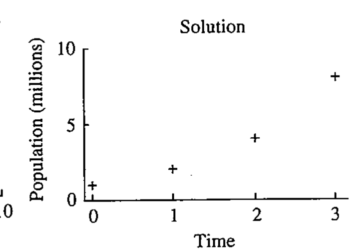
57.
a. $b_1 = 2b_0 - 1.0 \times 10^6 = 2 \cdot 3.0 \times 10^6 - 1.0 \times 10^6 = 5.0 \times 10^6$, $b_2 = 9.0 \times 10^6$, $b_3 = 17.0 \times 10^6$.
b. There were three harvests of 1.0×10^6 bacteria, for a total of 3.0×10^6 bacteria.
c. $b_{t+1} = 2.0 b_t - 1.0 \times 10^6$.
d. The population would have doubled three times, so there would be 24.0×10^6 bacteria. You could harvest 7.0×10^6 bacteria and still have a population of 17.0×10^6, which means harvesting 4.0×10^6 more than in part **b.** The bacteria removed early never had a chance to reproduce.

61. We have that $b_{t+1} = 2b_t$ and $m_{t+1} = 3m_t$. Then the total mass $M_{t+1} = m_{t+1} b_{t+1} = 3m_t \cdot 2b_t = 6 m_t b_t = 6 M_t$.

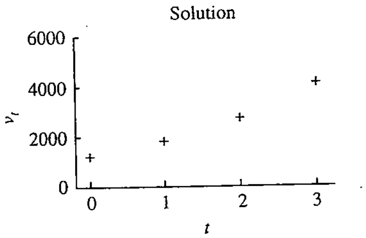
Section 1.6, page 76

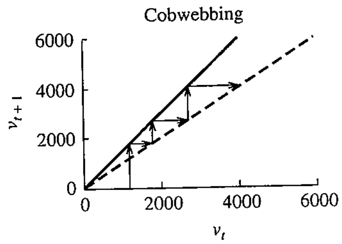
1.





3. The solution was $v_t = 1.5^t \cdot 1220\ \mu m^3$, consistent with a cobweb diagram that predicts a solution that increases faster and faster.





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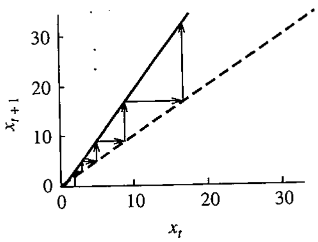

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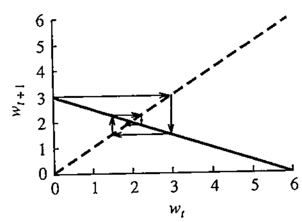
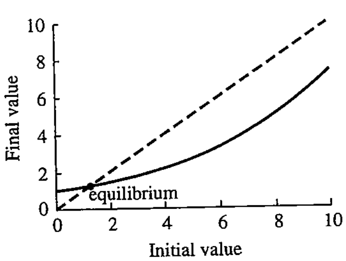
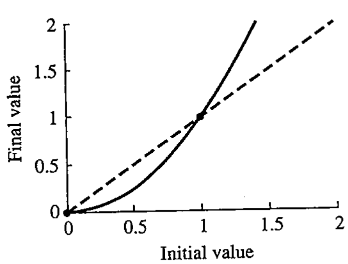


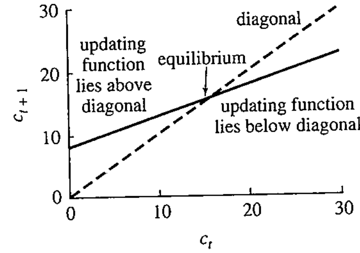
13. The equilibrium seems to be at about 1.3.



17. $f(x) = x$ when $x^2 = x$, or $x^2 - x = 0$, or $x(x-1) = 0$, which has solutions at $x = 0$ and $x = 1$.



19. The equilibrium is where $c^* = 0.5c^* + 8.0$, or $c^* = 16.0$.



23. $v^* = 1.5v^*$ if $v^* = 0$.

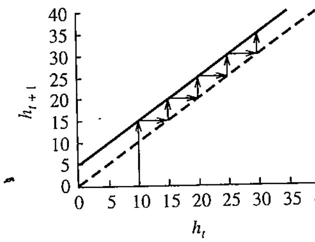
29. $x^* = \dfrac{x^*}{1+x^*}$ has solution $x^* = 0$.

31.

$$
\begin{aligned}
w^* &= aw^* + 3 \\
w^* - aw^* &= 3 \\
w^* &= \frac{3}{1-a}.
\end{aligned}
$$

This solution does not exist if $a = 1$, and it is negative if $a > 1$.

35.



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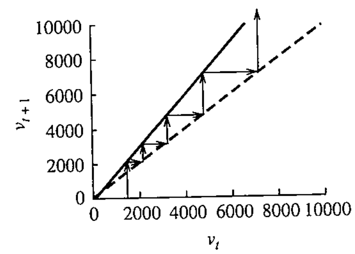

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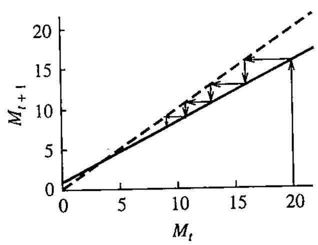
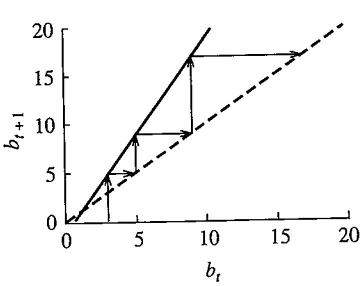


The equilibrium is

$$
\begin{aligned}
M^* &= 0.75M^* + 1 \\
M^* - 0.75M^* &= 1 \\
0.25M^* &= 1 \\
M^* &= 4.
\end{aligned}
$$

43.



The equilibrium is

$$
\begin{aligned}
b^* &= 2.0b^* - 1.0 \times 10^6 \\
b^* - 2.0b^* &= -1.0 \times 10^6 \\
-b^* &= -1.0 \times 10^6 \\
b^* &= 1.0 \times 10^6.
\end{aligned}
$$

The population grows, as we found in Section 1.5, Exercise 57, and seems to be moving away from the equilibrium.

45. The equilibrium is

$$
\begin{aligned}
b^* &= 2.0b^* - h \\
b^* - 2.0b^* &= -h \\
-b^* &= -h \\
b^* &= h.
\end{aligned}
$$

It is strange that the equilibrium gets larger as the harvest gets larger. However, the cobwebbing diagram indicates that only populations above the equilibrium will grow, and those below it will shrink. The equilibrium in this case is the minimum population required for the population to survive.

Section 1.7, page 89

1. Law 6: $43.2^0 = 1$.

3. Law 3: $43.2^{-1} = 1/43.2 \approx 0.023$.

5. Law 4: $43.2^{7.2}/43.2^{6.2} = 43.2^{7.2-6.2} = 43.2^1 = 43.2$.

7. Law 2: $(3^4)^{0.5} = 3^{4 \cdot 0.5} = 3^2 = 9$.

9. To use law 1, we must first multiply out the exponents, finding $2^{2^3} \cdot 2^{2^2} = 2^8 \cdot 2^4 = 2^{12} = 4096$.

11. Law 6: $\ln(1) = 0$.

13. Law 5: $\log_{43.2} 43.2 = 1$.

15. Law 1: $\log_{10}(5) + \log_{10}(20) = \log_{10}(5 \cdot 20) = \log_{10}(100) = \log_{10}(10^2) = 2$.

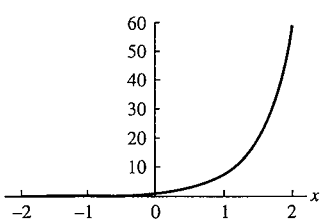
17. Law 4: $\log_{10}(500) - \log_{10}(50) = \log_{10}(500/50) = \log_{10}(10) = 1$.

19. Law 2: $\log_{43.2}(43.2^7) = 7$.

21. Law 3: $\log_7(1/43.2) = -\log_7 43.2 \approx -1.935$.

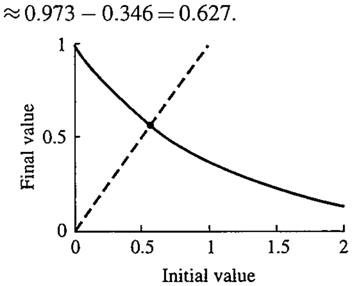
23. $e^{3x} = \frac{21}{7} = 3$. Taking logs of both sides, $3x = \ln(3) \approx 1.099$, and $x \approx 0.366$. Checking, $7e^{3 \cdot 0.366} \approx 21.0$.

27. $e^{2x} = 7$ at $2x = \ln(7.0) \approx 1.946$ or $x \approx 0.973$. This function is increasing, and it doubles after a "time" of $x = \frac{\ln(2)}{2} \approx 0.346$.

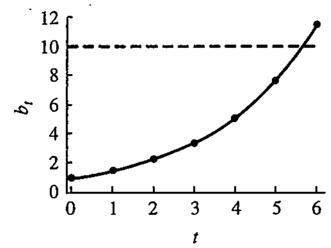


Therefore, $e^{2x} = 14.0$ when $x \approx 0.973 + 0.346 = 1.319$, and $e^{2x} = 3.5$ when $x \approx 0.973 - 0.346 = 0.627$.

31.



33. The solution is $b_t = 1.5^t \cdot 1.0 \times 10^6$. In exponential notation, $b_t = 1.0 \times 10^6 e^{\ln(1.5)t} \approx 1.0 \times 10^6 e^{0.405t}$. The population reaches 1.0×10^7 when $e^{0.405t} = 10$, or $0.405t = \ln(10) \approx 2.302$, or $t \approx 2.302/0.405 \approx 5.865$.



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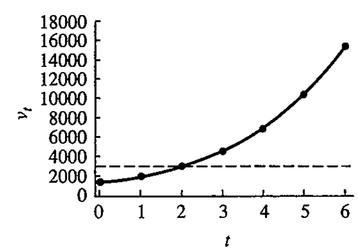


37. $t_d \approx 0.6931/1.0 = 0.6931$ days. It will take twice this long to quadruple, or 1.386 days.

39. $t_d \approx 0.6931/0.1 = 6.931$ hr. It will quadruple in 13.86 hr.

41. Converting to base e, we have that $S(t) = 2.34e^{\ln(10) \cdot 0.5t} \approx 2.34e^{1.151t}$. The doubling time is $t_d \approx 0.6931/1.151 = 0.602$ days. It will have increased by a factor of 10 when $\alpha t = 1$, which occurs when $t = 2$.

43. $Q(50000) = 6.0 \times 10^{10} e^{-0.000122 \cdot 50000} \approx 1.34 \times 10^8$. This is $1.34 \times 10^8/6.0 \times 10^{10} \approx 0.00223$ of the original amount.

45. $t_h \approx 0.693/0.000122 \approx 5680$ years.

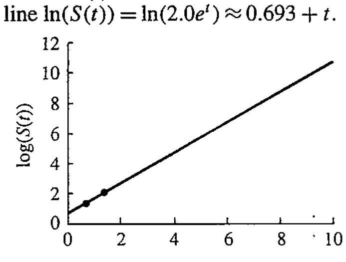
47. It will have doubled twice and will be 2000.

49. If the doubling time is 24 years, the parameter α is $0.693/24 \approx 0.0289$. Therefore, $P(t) \approx 500e^{0.0289t}$.

51. It will have halved three times, which takes 129 years.

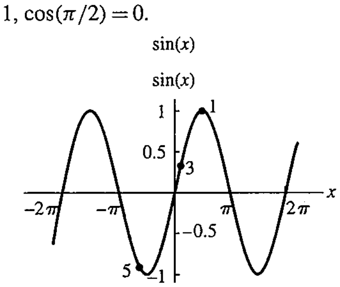
53. If the half-life is 43 years, the parameter α is $-0.693/43 \approx -0.0161$, and the equation is $P(t) \approx 1600e^{-0.0161t}$.

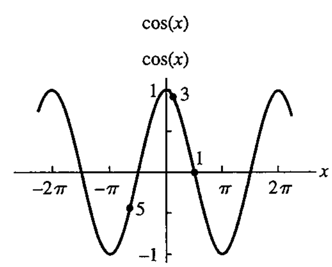
55. We plot the line $\ln(S(t)) = \ln(2.0e^t) \approx 0.693 + t$.

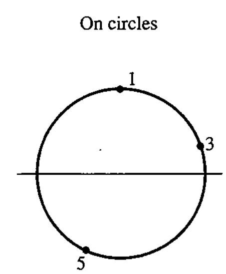


Section 1.8, page 98

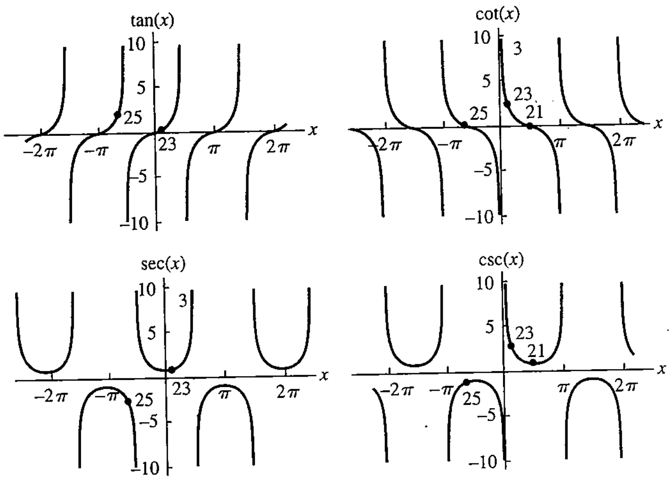
1. $\sin(\pi/2) = 1$, $\cos(\pi/2) = 0$.



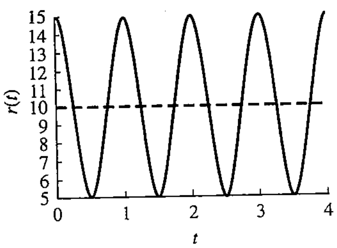




3. $\sin(\pi/9) \approx 0.342$, $\cos(\pi/9) \approx 0.940$.
5. $\sin(-2.0) \approx -0.909$, $\cos(-2.0) \approx -0.416$.
7. $\pi/6$ rad.
9. 0.017 rad.
11. 114.6°.
13. $-36° = 324°$.
15. 200 grads, after multiplying by 400 grads/360°.
19. 135°, after multiplying by 360°/400 grads.
21. No answer, 0, no answer, 1.

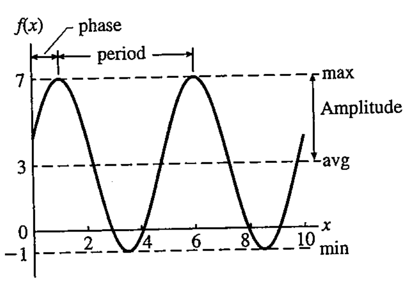


23. $\tan(\pi/9) \approx 0.36397$, $\cot(\pi/9) \approx 2.74748$, $\sec(\pi/9) \approx 1.06418$, $\csc(\pi/9) \approx 2.92380$.
25. $\tan(-2.0) \approx 2.18504$, $\cot(-2.0) \approx 0.45766$, $\sec(-2.0) \approx -2.40300$, $\csc(-2.0) \approx -1.09975$.
27. $\cos(0) = \sqrt{1}$, $\cos(\pi/4) = \sqrt{2}/2 = \sqrt{1/2}$, $\cos(\pi/2) = 0 = \sqrt{0/2}$.
33. Multiplying the factor of 5.0 through gives $r(t) = 10.0 + 5.0\cos(2\pi t)$ with average 10.0, amplitude 5.0, period 1.0, and phase 0.

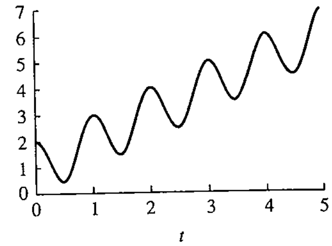


37. Average is 6.0, minimum is 4.0, maximum is 8.0, amplitude is 2.0, period is 5.0, and phase is 2.0.
41. The average is 3.0, the amplitude is 4.0, the maximum is 7.0, the minimum is −1.0, the period is 5.0, and the phase is 1.0.

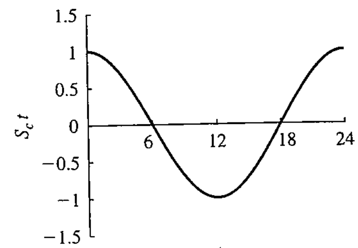
Answer to part a



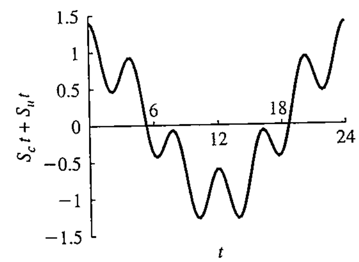
45. This function increases overall but wiggles around quite a bit. It might describe the size of an organism that grows on average, but grows quickly during the day and shrinks down a bit at night.



51. Let $S_c(t)$ represent the circadian rhythm. Then $S_c(t) = \cos\left(\frac{2\pi t}{24}\right)$.



53. To plot, compute the value of the combined function $S_c(t) + S_u(t)$ every hour, and smoothly connect the dots.



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1. 2/3 of the water is at 100°C, and 1/3 is at 30°C. The final temperature is then the weighted average $T = 1/3 \cdot 30°\text{C} + 2/3 \cdot 100°\text{C} \approx 76.7°\text{C}$.
5. 1/3 of the water is at T_1, and 2/3 is at T_2. The final temperature is then the weighted average $T = 1/3 \cdot T_1 + 2/3 \cdot T_2$. If $T_1 = 30$ and $T_2 = 100$, we get $T = 1/3 \cdot 30 + 2/3 \cdot 100 = 76.7$ as before.
7. A fraction $V_1/(V_1 + V_2)$ is at T_1, and a fraction $V_2/(V_1 + V_2)$ is at T_2. The final temperature is the weighted average

$$T = T_1 \frac{V_1}{V_1 + V_2} + T_2 \frac{V_2}{V_1 + V_2}.$$

9. The 100°C water cools to 50°C, so 2/3 of the water is at 50°C, and 1/3 is at 15°C. The final temperature is then the weighted average $T = 1/3 \cdot 15°\text{C} + 2/3 \cdot 50°\text{C} \approx 38.3°\text{C}$. This is indeed half the value in Exercise 1.
13.
a. amount = volume times concentration, or $V \cdot c_0 = 2.0 \cdot 1.0 = 2.0$ mmol.
b. 0.5 L at 1.0 mmol/L = 0.5 mmol.
c. 1.5 L at 1.0 mmol/L = 1.5 mmol.
d. 0.5 L at 5.0 mmol/L = 2.5 mmol.
e. 1.5 mmol + 2.5 mmol = 4.0 mmol.
f. 4.0 mmol/2.0 L = 2.0 mmol/L.
g. $q = 0.5/2.0 = 0.25$. Then $c_{t+1} = (1-q)c_t + q\gamma = 0.75 \cdot c_t + 0.25 \cdot 5.0$. When $c_0 = 1.0$, $c_1 = 0.75 \cdot 1.0 + 0.25 \cdot 5.0 = 2.0$ mmol/L.

17. The discrete-time dynamical system has $q = 0.5/2.0 = 0.25$ and $\gamma = 5.0$ and thus has formula $c_{t+1} = (1 - 0.25)c_t + 0.25 \cdot 5.0 = 0.75c_t + 1.25$. We want to start from 1.0 mmol/L.