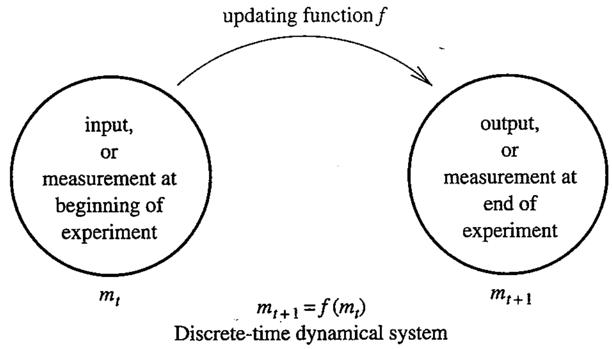



**FIGURE 1.5.66**
Notation for a discrete-time dynamical system

# 1.5 Discrete-Time Dynamical Systems

Suppose we collect data on how much several bacterial cultures grow in one hour or on how much trees grow in one year. How can we predict what will happen in the long run? In this section, we begin addressing these dynamical problems, which form the theme of this chapter and indeed of much of this book. We follow the basic steps of applied mathematics: **quantifying the basic measurement** and describing the **dynamical rule.** We will learn how to summarize the rule with a **discrete-time dynamical system** or an **updating function** that describes change. From the discrete-time dynamical system and a starting point, called an **initial condition,** we will compute a **solution** that gives the values of the measurement as a function of time.

## Discrete-Time Dynamical Systems and Updating Functions

A discrete-time dynamical system describes the relation between a quantity measured at the beginning and the end of an experiment or a time interval. If the measurement is represented by the variable $m$, we will use the notation $m_t$ to denote the measurement at the beginning of the experiment and $m_{t+1}$ to denote the measurement at the end of the experiment (Figure 1.5.66). Think of $t$ as the current time and of $t+1$ as the time one step into the future. The relation between the initial measurement $m_t$ and the final measurement $m_{t+1}$ is given by the **discrete-time dynamical system**

$$m_{t+1} = f(m_t) \qquad \textbf{(1.5.1)}$$

The **updating function** $f$ accepts the initial value $m_t$ as input and returns the final value $m_{t+1}$ as output.

We will begin by applying this notation to several examples of discrete-time dynamical systems.

**Example 1.5.1** A Discrete-Time Dynamical System for a Bacterial Population

Recall the data introduced in Example 1.2.3. Several bacterial cultures with different initial population sizes are grown in controlled conditions for 1 hour and then carefully measured.

| Colony | Initial Population, $b_t$ | Final Population, $b_{t+1}$ |
|---|---|---|
| 1 | 0.47 | 0.94 |
| 2 | 3.3 | 6.6 |
| 3 | 0.73 | 1.46 |
| 4 | 2.8 | 5.6 |
| 5 | 1.5 | 3.0 |
| 6 | 0.62 | 1.24 |

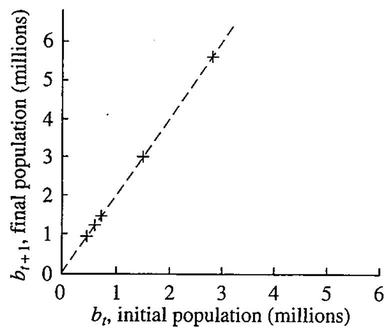



**FIGURE 1.5.67**
Graph of the updating function for a bacterial population

We have replaced $b_i$ (the initial population) with $b_t$ (the population at time $t$) and $b_f$ (the final population) with $b_{t+1}$ (the population at time $t + 1$).

In each colony, the population doubled in size. We can describe this with the discrete-time dynamical system

$$b_{t+1} = 2.0b_t$$

The updating function $f$ describes the rule applied to the initial population,

$$f(b_t) = 2.0b_t$$

As we have seen, a graph of the updating function plots the initial measurement $b_t$ on the horizontal axis and the final measurement $b_{t+1}$ on the vertical axis (Figure 1.5.67).

**Example 1.5.2** A Discrete-Time Dynamical System for Tree Growth

Suppose you measure the heights of several trees in one year and then again the next year. Denoting the initial height by $h_t$ and the final height by $h_{t+1}$, you might find the data in the following table (all expressed in meters).

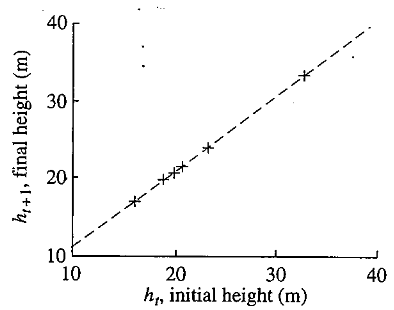



**FIGURE 1.5.68**
Data describing the growth of six trees

| Tree | Initial Height, $h_t$ | Final Height, $h_{t+1}$ | Change in Height |
|---|---|---|---|
| 1 | 23.1 | 24.1 | 1.0 |
| 2 | 18.7 | 19.8 | 1.1 |
| 3 | 20.6 | 21.5 | 0.9 |
| 4 | 16.0 | 17.0 | 1.0 |
| 5 | 32.5 | 33.6 | 1.1 |
| 6 | 19.8 | 20.6 | 0.8 |

The trees increase in height by about 1.0 m per year (Figure 1.5.68).

If we approximate this by assuming that trees grow exactly 1.0 m per year, then the discrete-time dynamical system that expresses this relation is

$$h_{t+1} = h_t + 1.0$$

The updating function, which we can denote by $g$, has formula

$$g(h_t) = h_t + 1.0$$

For example, for a tree beginning with height 12.2 m, the discrete-time dynamical system predicts a final height of

$$h_{t+1} = g(12.2) = 12.2 + 1.0 = 13.2\,\text{m}$$

In this example, the data points do not exactly match the discrete-time dynamical system. The updating function captures the major trend in the data while ignoring the noise. Including only the trend corresponds to the use of a **deterministic** dynamical system to describe the behavior. To include the noise, we must use a **probabilistic** dynamical system (Chapter 6). We will specifically address the problem of **finding** an updating function that captures the major trends in the data when we study the technique of data-fitting called linear regression (Section 8.9).

**Example 1.5.3** Discrete-Time Dynamical System for Mites

Recall the lizards infested by mites (Example 1.4.12). The final number of mites $x_{t+1}$ is related to the initial number of mites $x_t$ by the formula

$$x_{t+1} = 2x_t + 30$$

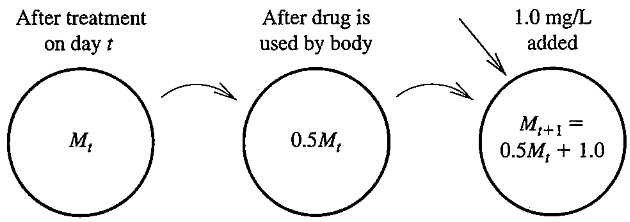



FIGURE 1.5.69
The dynamics of medication concentration in the blood

This is the discrete-time dynamical system for this population. The updating function is

$$h(x_t) = 2x_t + 30$$

The discrete-time dynamical systems for bacterial populations, tree height, and mite number were all derived from data. Often, dynamical rules can instead be derived directly from the principles governing a system.

**Example 1.5.4** A Discrete-Time Dynamical System for Medication Concentration

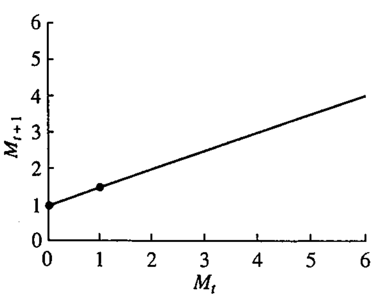



FIGURE 1.5.70
A graph of the updating function for medication concentration

Suppose we know the following facts about the dynamics of medication. Each day, a patient uses up half of the medication in his bloodstream. However, he is given a new dose sufficient to raise the concentration in the bloodstream by 1.0 milligram per liter (Figure 1.5.69). Let $M_t$ denote the concentration at time $t$. The discrete-time dynamical system is

$$M_{t+1} = 0.5M_t + 1.0$$

The term $0.5M_t$ indicates that only half of the initial medication **remains** the next day. The factor 0.5 is the slope of this linear function. The second term, the intercept, indicates that 1.0 milligram per liter of medication is added each day. We can graph this linear function by substituting two reasonable values for $M_t$. If $M_t = 0$, then $M_{t+1} = 1$, the vertical intercept of this line. If $M_t = 1$, then $M_{t+1} = 1.5$ (Figure 1.5.70).

## Manipulating Updating Functions

All of the operations that can be applied to ordinary functions can be applied to updating functions, but with special interpretations. We will study **composition** of an updating function with itself, find the **inverse** of an updating function, and **convert the units** or **translate the dimensions** of a discrete-time dynamical system.

**Composition** Consider the discrete-time dynamical system

$$m_{t+1} = f(m_t)$$

with updating function $f$. What does the composition $f \circ f$ mean? The updating function **updates** the measurement by one time step. Then

$$\begin{aligned} (f \circ f)(m_t) &= f(f(m_t)) && \text{definition of composition} \\ &= f(m_{t+1}) && \text{definition of updating function} \\ &= m_{t+2} && \text{updating function applied to } m_{t+1} \end{aligned}$$

Therefore,

$$(f \circ f)(m_t) = m_{t+2}$$

The composition of an updating function with itself corresponds to a two-step updating function (Figure 1.5.71).

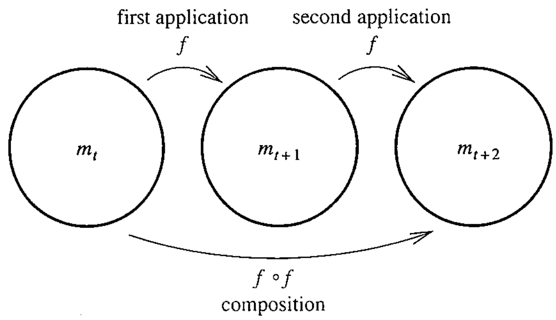



FIGURE 1.5.71
Composition of an updating function with itself

**Example 1.5.5** Composition of the Bacterial Population Updating Function with Itself

The bacterial updating function is $f(b_t) = 2b_t$. The function $f \circ f$ takes the population size at time $t$ as input and returns the population size 2 hours later, at time $t + 2$, as output. We can compute $f \circ f$ with the steps

$$\begin{aligned}(f \circ f)(b_t) &= f(f(b_t)) \\ &= f(2.0b_t) \\ &= 2.0 \times 2.0b_t \\ &= 4.0b_t\end{aligned}$$

After two hours, the population is four times larger, having doubled twice. In this case, composition of $f$ with itself looks like multiplication. This simple rules works **only** for an updating function expressing a proportional relation.

**Example 1.5.6** Composition of the Mite Population Updating Function with Itself

The composition of the mite population updating function $h(x_t) = 2x_t + 30$ with itself gives

$$\begin{aligned}(h \circ h)(x_t) &= h(h(x_t)) \\ &= h(2x_t + 30) \\ &= 2(2x_t + 30) + 30 \\ &= 4x_t + 90\end{aligned}$$

Suppose we started with $x_t = 10$ mites. After 1 week, we would find $h(10) = 2 \cdot 10 + 30 = 50$ mites. After a second week, we would find $h(50) = 2 \cdot 50 + 30 = 130$ mites. Using the composition of the updating function with itself, we can compute the number of mites after 2 weeks, skipping over the intermediate value of 50 mites after 1 week, finding

$$(h \circ h)(10) = 4 \cdot 10 + 90 = 130$$

**Inverses** Consider again the general discrete-time dynamical system

$$m_{t+1} = f(m_t)$$

with updating function $f$. What does the inverse $f^{-1}$ mean? The updating function updates the measurement by one time step, and the inverse function undoes the action of the updating function. Therefore,

$$f^{-1}(m_{t+1}) = m_t$$

The inverse of an updating function corresponds to an "updating" function that goes **backwards** in time (Figure 1.5.72).

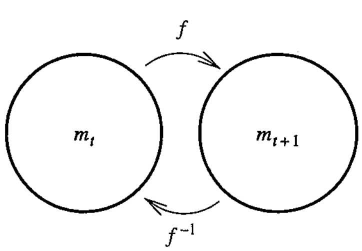



FIGURE 1.5.72
Inverse of an updating function

**Example 1.5.7** Inverse of the Bacterial Population Updating Function

The bacterial population updating function is $f(b_t) = 2b_t$. We find the inverse by writing the discrete-time dynamical system

$$b_{t+1} = 2.0b_t$$

and solving for the input variable $b_t$ (Algorithm 1.1). In this case, dividing both sides by 2.0 gives

$$b_t = \frac{b_{t+1}}{2.0}$$

The inverse function is

$$f^{-1}(b_{t+1}) = \frac{b_{t+1}}{2.0}$$

If **multiplying** by 2.0 describes how the population changes forward in time, **dividing** by 2.0 describes how it changes backwards in time.

If $b_t = 3.0$, then $b_{t+1} = 2.0b_t = 2.0 \cdot 3.0 = 6.0$. If we go backwards from $b_{t+1} = 6.0$ using the inverse of the updating function, we find

$$b_t = f^{-1}(6.0) = \frac{6.0}{3.0} = 3.0$$

exactly where we started.

**Example 1.5.8** Inverse of the Mite Population Updating Function

To find the inverse of the mite population updating function $h(x_t) = 2.0x_t + 30$, we use Algorithm 1.1

| | |
|---|---|
| $2.0x_t + 30 = x_{t+1}$ | the original equation |
| $2.0x_t = x_{t+1} - 30$ | subtract 30 from both sides |
| $x_t = \dfrac{x_{t+1} - 30}{2.0}$ | divide both sides by 2.0 |

Therefore,

$$x_t = h^{-1}(x_{t+1}) = \frac{x_{t+1} - 30}{2.0} = 0.5x_{t+1} - 15$$

Suppose we started with $x_t = 10$ mites. After one week, we would find

$$h(10) = 2 \cdot 10 + 30 = 50$$

Applying the inverse, we find

$$h^{-1}(50) = 0.5 \cdot 50 - 15 = 10$$

The inverse function takes us back to where we started.

## Discrete-Time Dynamical Systems: Units and Dimensions

The updating function $f(b_t) = 2.0b_t$ accepts as input positive numbers with units of bacteria. If we measure this quantity in different units, we must convert the updating function itself into the new units. If we measure a different quantity, such as total mass or volume, we can translate the updating function into different dimensions.

### Example 1.5.9 Describing the Dynamics of Tree Height in Centimeters

Suppose we wish to study tree height (Example 1.5.2) in units of centimeters rather than meters. In meters, the discrete-time dynamical system is

$$g(h_t) = h_t + 1.0\,\text{m}$$

First, we define a new variable to represent the measurement in the new units. Let $H_t$ be tree height measured in centimeters rather than meters. Then $H_t = 100h_t$, because there are 100 centimeters in a meter. We wish to find a discrete-time dynamical system that gives a formula for $H_{t+1}$ in terms of $H_t$ (Figure 1.5.73).

$$\begin{aligned} H_{t+1} &= 100h_{t+1} && \text{definition of } H_{t+1} \\ &= 100(h_t + 1.0) && \text{discrete-time dynamical system for } h_{t+1} \\ &= 100h_t + 100 && \text{multiply through by 100} \\ &= H_t + 100 && \text{definition of } H_t \end{aligned}$$

The discrete-time dynamical system in the new units corresponds to adding 100 centimeters to the height, which is equivalent to adding 1 meter. Although the underlying process is the same, the discrete-time dynamical system and the corresponding updating function are different, just as the numerical values of measurements are different in different units.

### Example 1.5.10 Describing the Dynamics of Bacterial Mass

Suppose we wish to study the bacterial population in terms of mass rather than number. At the beginning, the mass, denoted by $m_t$, is

$$m_t = \mu b_t$$

where $\mu$ is the mass per bacterium (as in Example 1.3.4). The updated mass $m_{t+1}$ is

$$\begin{aligned} m_{t+1} &= \mu b_{t+1} && \text{definition of } m_{t+1} \\ &= \mu \cdot 2.0b_t && \text{substitute the original updating function} \\ &= 2.0\mu b_t && \text{rearrange the terms by the associative and commutative laws} \\ &= 2.0m_t && \text{recognize that } m_t = \mu b_t \end{aligned}$$

This new discrete-time dynamical system doubles its input just as the original discrete-time dynamical system did, but it takes mass as its input rather than numbers of bacteria (Figure 1.5.74).

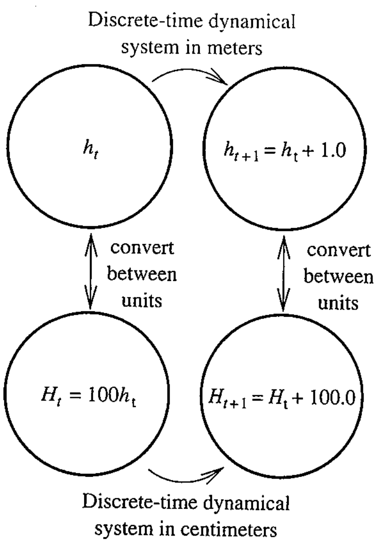



**FIGURE 1.5.73**
Finding the discrete-time dynamical system for trees in centimeters

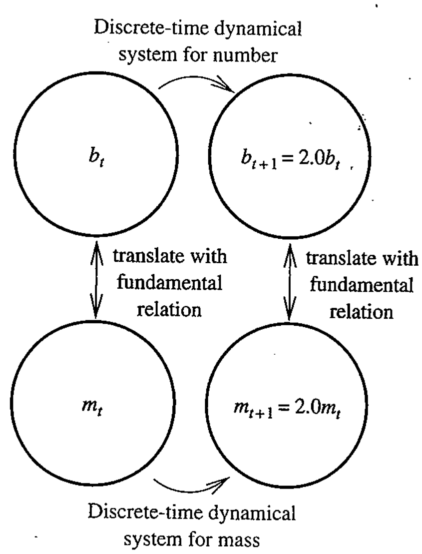



**FIGURE 1.5.74**
Finding the discrete-time dynamical system for bacteria in terms of mass

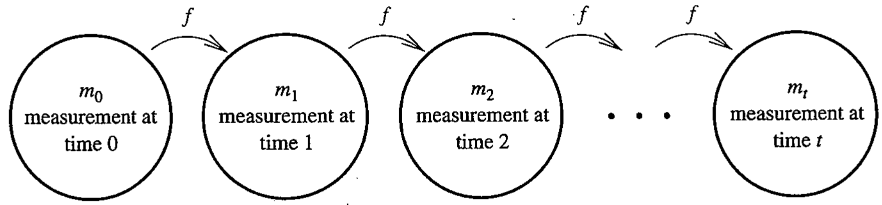



**FIGURE 1.5.75**
The repeated action of an updating function

## Solutions

A discrete-time dynamical system describes some quantity at the end of an experiment as a function of that same quantity at the beginning. What if we were to continue the experiment? A bacterial population growing according to $b_{t+1} = 2.0b_t$ would double again and again. A tree growing according to $h_{t+1} = h_t + 1.0$ would add more and more meters to its height. An infested lizard would become even more heavily infested.

To describe a situation where a dynamical process is repeated many times, we let $m_0$ represent the measurement at the beginning, $m_1$ the measurement after one time step, $m_2$ the measurement after two time steps, and so forth (Figure 1.5.75). In general, we define

$$m_t = \text{measurement } t \text{ hours after the beginning of the experiment.}$$

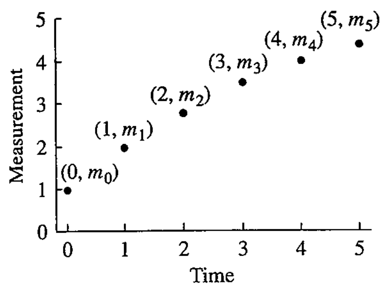



**FIGURE 1.5.76**
The graph of a solution

Our goal is to find the values of $m_t$ for all values of $t$. Before we can do so, however, we must know where we **started.** Without knowing where you started, it is impossible to answer a question such as "Where are you after driving 5 miles south?" The starting value is known as the **initial condition.**

**Definition 1.10** The sequence of values of $m_t$ for $t = 0, 1, 2, \ldots$ is the **solution** of the discrete-time dynamical system $m_{t+1} = f(m_t)$ starting from the **initial condition** $m_0$.

The graph of a solution is a discrete set of points with the time $t$ on the horizontal axis and the measurement $m_t$ on the vertical axis. The initial point has coordinates $(0, m_0)$ to describe the initial condition. The next point, with coordinates $(1, m_1)$, describes the measurement at $t = 1$, and so forth (Figure 1.5.76). It is possible to find a formula for the solution for simple discrete-time dynamical systems, but not in many more complicated cases.

**Example 1.5.11** A Solution of the Bacterial Discrete-time Dynamical System

Suppose we begin with one million bacteria, which corresponds to an initial condition of $b_0 = 1.0$ (with bacterial population measured in millions). If the bacteria obey the discrete-time dynamical system $b_{t+1} = 2.0b_t$, then

$$b_1 = 2.0b_0 = 2.0 \cdot 1.0 = 2.0$$
$$b_2 = 2.0b_1 = 2.0 \cdot 2.0 = 4.0$$
$$b_3 = 2.0b_2 = 2.0 \cdot 4.0 = 8.0$$

Examining these results, we notice that

$$b_1 = 2.0 \cdot 1.0$$
$$b_2 = 2.0^2 \cdot 1.0$$
$$b_3 = 2.0^3 \cdot 1.0$$

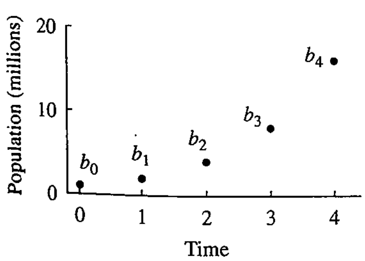



**FIGURE 1.5.77**
A solution: Bacterial population size as a function of time

After 3 hours, the population has doubled three times and is $2.0^3 = 8.0$ times the original population. We graph the solution by plotting the time $t$ on the horizontal axis and the number of bacteria after $t$ hours ($b_t$) on the vertical axis (Figure 1.5.77). The graph consists only of a discrete set of points describing the hourly measurements—hence the name *discrete-time dynamical system.* Sometimes, we will connect the points in a solution with line segments to make the pattern easier to see.

After $t$ hours, the population will have doubled $t$ times and will have reached the size

$$b_t = 2.0^t \cdot 1.0 \tag{1.5.2}$$

This formula describes the solution of the discrete-time dynamical system with initial condition $b_0 = 1.0$. It predicts the population after $t$ hours of reproduction for any value of $t$. For example, we can compute

$$b_8 = 2.0^8 \cdot 1.0 = 256.0$$

without ever computing $b_1$, $b_2$, or other intermediate values.

**Example 1.5.12** A Solution with a Different Initial Condition

Suppose we started the system with a different initial condition of $b_0 = 0.3$. We can find subsequent values by repeatedly applying the discrete-time dynamical system,

$$b_1 = 2.0 \cdot 0.3 = 0.6$$
$$b_2 = 2.0 \cdot 0.6 = 1.2$$
$$b_3 = 2.0 \cdot 1.2 = 2.4$$

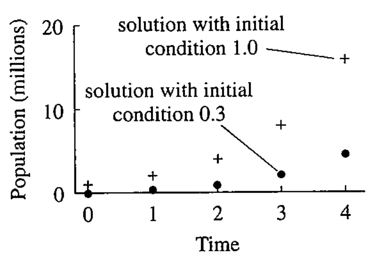



FIGURE 1.5.78
Solutions starting from two different initial conditions

If we look for the pattern in this case,

$$b_1 = 2.0 \cdot 0.3$$
$$b_2 = 2.0^2 \cdot 0.3$$
$$b_3 = 2.0^3 \cdot 0.3$$

After $t$ hours, the population will have doubled $t$ times, as before, and will have reached the size

$$b_t = 2.0^t \cdot 0.3 \text{ million bacteria}$$

The solution is **different** from the one found in Example 1.5.11 with a different initial condition (Figure 1.5.78). Although the two solutions get further and further apart, the ratio always remains the same (see Exercise 55, page 66).

**Example 1.5.13** Two Solutions of the Tree Height Discrete-time Dynamical System

Tree height obeys the discrete-time dynamical system

$$h_{t+1} = h_t + 1.0$$

(Example 1.5.2). Suppose the tree begins with a height of $h_0 = 10.0$ m. Then

$$h_1 = h_0 + 1.0 = 11.0 \text{ m}$$
$$h_2 = h_1 + 1.0 = 12.0 \text{ m}$$
$$h_3 = h_2 + 1.0 = 13.0 \text{ m}$$

Each year, the height of the tree increases by 1.0 m. After 3 years, the height is 3.0 m greater than the original height. After $t$ years the tree has added 1.0 m to its height $t$ times, meaning that the height will have increased by a total of $t$ m. Therefore, the solution is

$$h_t = 10.0 + t \text{ m}$$

This formula predicts the height after $t$ years of growth for any $t$. We can compute

$$h_8 = 10.0 + 8.0 = 18.0 \text{ m}$$

without computing $h_1$, $h_2$, or other intermediate values (Figure 1.5.79).

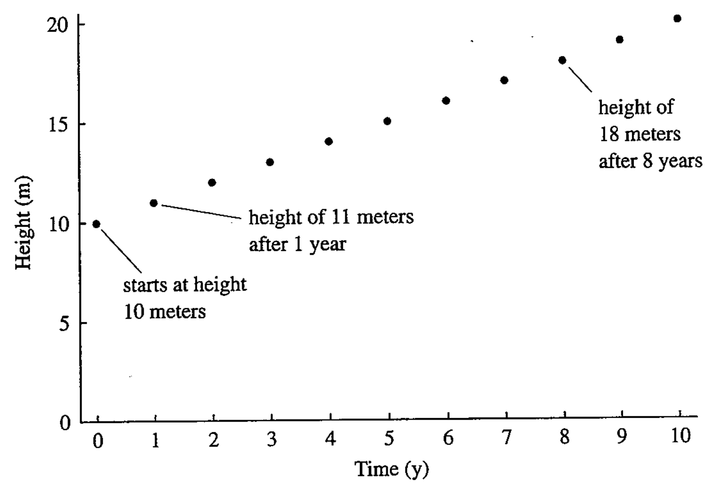



**FIGURE 1.5.79**
A solution: tree height as a function of time

If the tree began at the smaller size of 2.0 m, the size for the first few years would be

$$h_1 = h_0 + 1.0 = 3.0 \text{ m}$$
$$h_2 = h_1 + 1.0 = 4.0 \text{ m}$$
$$h_3 = h_2 + 1.0 = 5.0 \text{ m}$$

Again, the tree adds $t$ m of height in $t$ years, so the height is

$$h_t = 2.0 + t \text{ m}$$

The solution with this smaller initial condition is always exactly 8.0 m less than the solution found before (Figure 1.5.80).

Is it always possible to guess the formula for a solution in this way? We will next see some cases where computing the solution step by step is straightforward but finding a formula for the solution is tricky. Remarkably, there are simple discrete-time dynamical systems for which it is *impossible* to write a formula for a solution. For example, **chaotic** dynamical systems have solutions so unpredictable that no formula can describe them. (See "Analysis of the Logistic Dynamical System," p. 257–261, in Section 3.2.)

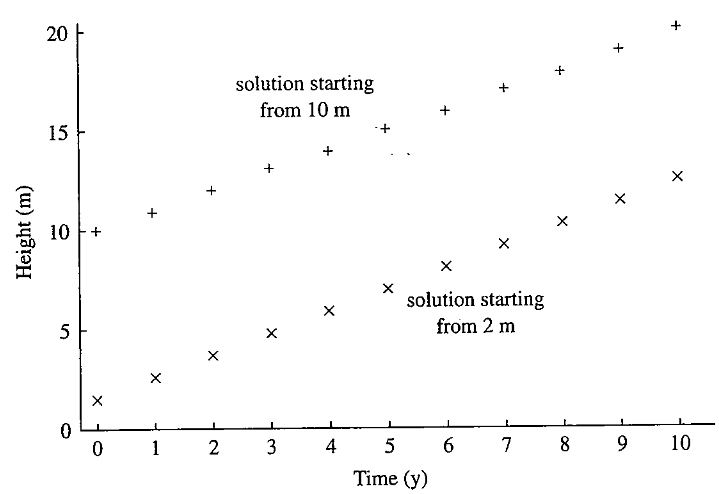



**FIGURE 1.5.80**
Two solutions for tree height as functions of time

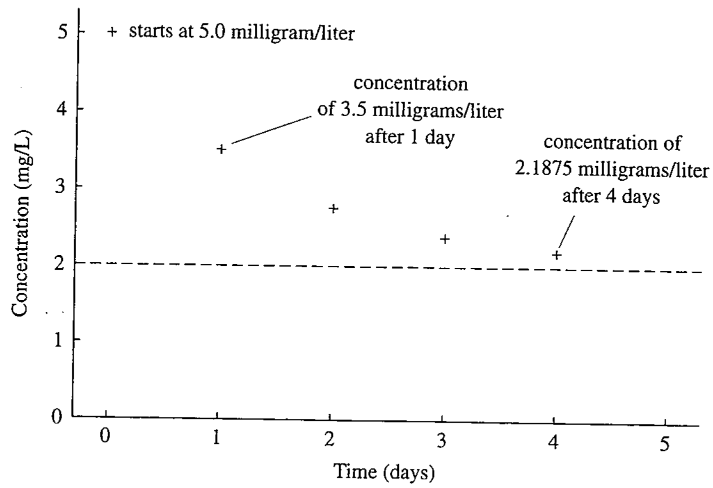



**FIGURE 1.5.81**
Medication concentration as a function of time

**Example 1.5.14** Finding a Solution of the Medication Discrete-time Dynamical System

Consider the discrete-time dynamical system for medication (Example 1.5.4) given by

$$M_{t+1} = 0.5M_t + 1.0$$

Suppose we begin from an initial condition of $M_0 = 5.0$ milligrams per liter. Then

$$\begin{aligned}
M_1 &= 0.5 \cdot 5.0 + 1.0 = 3.5 \\
M_2 &= 0.5 \cdot 3.5 + 1.0 = 2.75 \\
M_3 &= 0.5 \cdot 2.75 + 1.0 = 2.375 \\
M_4 &= 0.5 \cdot 2.375 + 1.0 = 2.1875
\end{aligned}$$

The values are getting closer and closer to 2.0 (Figure 1.5.81). More careful examination indicates that the results move exactly **halfway** toward 2.0 each step. In particular, we find that the difference between the measured rate and 2.0 is

$$\begin{aligned}
M_0 - 2.0 &= 5.0 - 2.0 = 3.0 \\
M_1 - 2.0 &= 3.5 - 2.0 = 1.5 = 0.5 \cdot 3.0 \\
M_2 - 2.0 &= 2.75 - 2.0 = 0.75 = 0.5 \cdot 1.5 \\
M_3 - 2.0 &= 2.375 - 2.0 = 0.375 = 0.5 \cdot 0.75 \\
M_4 - 2.0 &= 2.1875 - 2.0 = 0.1875 = 0.5 \cdot 0.375
\end{aligned}$$

Can we convert these observations into the formula for a solution? If we write the concentration as 2.0 plus the difference,

$$\begin{aligned}
M_0 &= 2.0 + 3.0 \\
M_1 &= 2.0 + 0.5 \cdot 3.0 \\
M_2 &= 2.0 + 0.5^2 \cdot 3.0 \\
M_3 &= 2.0 + 0.5^3 \cdot 3.0
\end{aligned}$$

we might see that

$$M_t = 2.0 + 0.5^t \cdot 3.0$$

Finding patterns in this way and translating them into formulas can be tricky. It is much more important to be able to *describe* the behavior of solutions with a graph or in words. In this case, our calculations quickly revealed that the solution moved closer and closer to 2.0. In Section 1.6, we will develop a powerful graphical method to deduce this pattern with a minimum of calculation. ▲

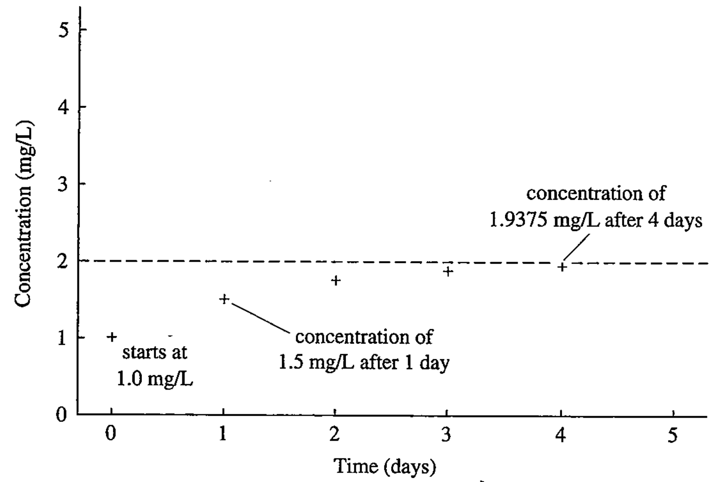



**FIGURE 1.5.82**
Another solution of medication concentration as a function of time

**Example 1.5.15** A Second Solution of the Medication Discrete-time Dynamical System

If we begin with an initial concentration of $M_0 = 1.0$ milligrams per liter, then

$$\begin{aligned} M_1 &= 0.5 \cdot 1.0 + 1.0 = 1.5 \\ M_2 &= 0.5 \cdot 1.5 + 1.0 = 1.75 \\ M_3 &= 0.5 \cdot 1.75 + 1.0 = 1.875 \\ M_4 &= 0.5 \cdot 1.875 + 1.0 = 1.9375 \end{aligned}$$

(See Figure 1.5.82.) Unlike graphs of bacterial populations (Example 1.5.12) and tree size (Example 1.5.13), the graphs of solutions starting from different initial conditions look completely different.

However, the values still get closer and closer to 2.0, and the difference from 2.0 is reduced by a factor of 2 each day:

$$\begin{aligned} M_0 - 2.0 &= 1.0 - 2.0 = -1.0 \\ M_1 - 2.0 &= 1.5 - 2.0 = -0.5 \\ M_2 - 2.0 &= 1.75 - 2.0 = -0.25 \\ M_3 - 2.0 &= 1.875 - 2.0 = -0.125 \\ M_4 - 2.0 &= 1.9375 - 2.0 = -0.0625 \end{aligned}$$

We can find the formula using the same idea as before. If we write

$$\begin{aligned} M_0 &= 2.0 - 1.0 \\ M_1 &= 2.0 - 0.5 \cdot 1.0 \\ M_2 &= 2.0 - 0.5^2 \cdot 1.0 \\ M_3 &= 2.0 - 0.5^3 \cdot 1.0 \end{aligned}$$

we can see that

$$M_t = 2.0 - 0.5^t \cdot 1.0$$

In Section 2.2, we will use the fundamental idea of the **limit** to study more carefully what it means for the sequence of values that define a solution to get closer and closer to 2.0.

**Example 1.5.16** A Solution of the Mite Population Discrete-time Dynamical System

Recall the discrete-time dynamical system

$$x_{t+1} = 2x_t + 30$$

for mites. If we started our lizard off with $x_0 = 10$ mites, we compute

$$x_1 = 2.0x_0 + 30 = 50$$
$$x_2 = 2.0x_1 + 30 = 130$$
$$x_3 = 2.0x_2 + 30 = 290$$

The pattern is not too obvious in this case. There is a pattern, however, which it is a good challenge to find (Exercise 35, p. 65).

**Summary** Starting from data or an understanding of a biological process, we can derive a **discrete-time dynamical system,** the **dynamical rule** that tells how a measurement changes from one time step to the next. The **updating function** describes the relation between measurements at times $t$ and $t+1$. The **composition** of the updating function with itself produces a two-step discrete-time dynamical system, and the **inverse** of the updating function produces a backwards discrete-time dynamical system. Like all biological relations, a discrete-time dynamical system can be described in different units and dimensions. Repeated application of a discrete-time dynamical system starting from an **initial condition** generates a **solution,** the value of the measurement as a function of time. With the proper combination of diligence, cleverness, and luck, it is sometimes possible to find a formula for the solution.

## 1.5 Exercises

### Mathematical Techniques

**1–4** ▪ Write the updating function associated with each of the following discrete-time dynamical systems and evaluate it at the given arguments. Which are linear?

1. $p_{t+1} = p_t - 2$, evaluate at $p_t = 5$, $p_t = 10$, and $p_t = 15$.
2. $\psi_{t+1} = \frac{\psi_t}{2}$, evaluate at $\psi_t = 4$, $\psi_t = 8$, and $\psi_t = 12$.
3. $x_{t+1} = x_t^2 + 2$, evaluate at $x_t = 0$, $x_t = 2$, and $x_t = 4$.
4. $Q_{t+1} = \frac{1}{Q_t + 1}$, evaluate at $Q_t = 0$, $Q_t = 1$, and $Q_t = 2$.

**5–8** ▪ Compose with itself the updating function associated with each discrete-time dynamical system. Find the two-step discrete-time dynamical system. Check that the result of applying the original discrete-time dynamical system to the given initial condition twice matches the result of applying the new discrete-time dynamical system to the given initial condition once.

5. Volume follows $v_{t+1} = 1.5v_t$, starting from $v_0 = 1220\,\mu\text{m}^3$.
6. Length obeys $l_{t+1} = l_t - 1.7$, starting from $l_0 = 13.1$ cm.
7. Population size follows $n_{t+1} = 0.5n_t$, starting from $n_0 = 1200$.
8. Medication concentration obeys $M_{t+1} = 0.75M_t + 2.0$ starting from the initial condition $M_0 = 16.0$.

**9–12** ▪ Find the backwards discrete-time dynamical system associated with each discrete-time dynamical system. Use it to find the value at the previous time.

9. $v_{t+1} = 1.5v_t$. Find $v_0$ if $v_1 = 1220\,\mu\text{m}^3$.
10. $l_{t+1} = l_t - 1.7$. Find $l_0$ if $l_1 = 13.1$ cm.
11. $n_{t+1} = 0.5n_t$. Find $n_0$ if $n_1 = 1200$.
12. $M_{t+1} = 0.75M_t + 2.0$. Find $M_0$ if $M_1 = 16.0$.

**13–14** ▪ Find the composition of each of the following mathematically elegant updating functions with itself, and find the inverse function.

13. The updating function $f(x) = \frac{x}{1+x}$. Remember to put things over a common denominator to simplify the composition.
14. The updating function $h(x) = \frac{x}{x-1}$. Remember to put things over a common denominator to simplify the composition.

**15–18** ▪ Find and graph the first five values of the following discrete-time dynamical systems, starting from the given initial condition. Compare the graph of the solution with the graph of the updating function.

15. $v_{t+1} = 1.5v_t$, starting from $v_0 = 1220\,\mu\text{m}^3$.
16. $l_{t+1} = l_t - 1.7$, starting from $l_0 = 13.1$ cm.
17. $n_{t+1} = 0.5n_t$, starting from $n_0 = 1200$.
18. $M_{t+1} = 0.75M_t + 2.0$ starting from the initial condition $M_0 = 16.0$.

**19–22** ▪ Using a formula for the solution, you can project far into the future without computing all the intermediate values. Find the following, and indicate whether the results are reasonable.

19. From the solution found in Exercise 15, find the volume at $t = 20$.
20. From the solution found in Exercise 16, find the length at $t = 20$.

**21.** From the solution found in Exercise 17, find the number at $t = 20$.

**22.** From the solution found in Exercise 18, find the concentration at $t = 20$.

**23–26** ▪ Experiment with the following mathematically elegant updating functions and try to find the solution.

**23.** Consider the updating function

$$f(x) = \frac{x}{1+x}$$

from Exercise 13. Starting from an initial condition of $x_0 = 1$, compute $x_1$, $x_2$, $x_3$, and $x_4$, and try to spot the pattern.

**24.** Use the updating function in Exercise 23, but start from the initial condition $x_0 = 2$.

**25.** Consider the updating function

$$g(x) = 4 - x$$

Start from initial condition of $x_0 = 1$, and try to spot the pattern. Experiment with a couple of other initial conditions. How would you describe your results in words?

**26.** Consider the updating function

$$h(x) = \frac{x}{x-1}$$

from Exercise 14. Start from initial condition of $x_0 = 3$, and try to spot the pattern. Experiment with a couple of other initial conditions. How would you describe your results in words?

## Applications

**27–30** ▪ Consider the following actions. Which of them commute (produce the same answer when done in either order)?

**27.** A population doubles in size; 10 individuals are removed from a population. Try starting with 100 individuals, and then try to figure out what happens in general.

**28.** A population doubles in size; population size is divided by 4. Try starting with 100 individuals, and then try to figure out what happens in general.

**29.** An organism grows by 2.0 cm; an organism shrinks by 1.0 cm.

**30.** A person loses half her money. A person gains \$10.

**31–34** ▪ Use the formula for the solution to find the following, and indicate whether the results are reasonable.

**31.** Using the solution for tree height $h_t = 10.0 + t$ (Example 1.5.13), find the tree height after 20 years.

**32.** Using the solution for tree height $h_t = 10.0 + t$ (Example 1.5.13), find the tree height after 100 years.

**33.** Using the solution for bacterial population number $b_t = 2.0^t \cdot 1.0$ (Equation 1.5.2), find the bacterial population after 20 hours. If an individual bacterium weighs about $10^{-12}$ g, how much will the whole population weigh?

**34.** Using the solution for bacterial population number $b_t = 2.0^t \cdot 1.0$ (Equation 1.5.2), find the bacterial population after 40 hours.

**35–36** ▪ Try to find a formula for the solution of the given discrete-time dynamical system.

**35.** Find the pattern in the number of mites on a lizard, starting with $x_0 = 10$ and following the discrete-time dynamical system $x_{t+1} = 2x_t + 30$. (*Hint:* add 30 to the number of mites.)

**36.** Try to find the pattern in the number of mites on a lizard, starting with $x_0 = 10$ and following the discrete-time dynamical system $x_{t+1} = 2x_t + 20$.

**37–40** ▪ The following tables display data from four experiments:

1. Cell volume after 10 minutes in a watery bath
2. Fish length after 1 week in a chilly tank
3. Gnat population size after 3 days without food
4. Yield (in bushels) of several varieties of soybeans before and one month after fertilization.

For each, graph the new value as a function of the initial value, find a simple discrete-time dynamical system, and fill in the missing value in the table.

**37.**

| Cell Volume ($\mu m^3$) | |
|---|---|
| Initial, $v_t$ | Final, $v_t$ |
| 1220 | 1830 |
| 1860 | 2790 |
| 1080 | 1620 |
| 1640 | 2460 |
| 1540 | 2310 |
| 1420 | ?? |

**38.**

| Fish Mass (g) | |
|---|---|
| Initial, $m_t$ | Final, $m_{t+1}$ |
| 13.1 | 11.4 |
| 18.2 | 16.5 |
| 17.3 | 15.6 |
| 16.0 | 14.3 |
| 20.5 | 18.8 |
| 1.5 | ?? |

**39.**

| Gnat Number | |
|---|---|
| Initial, $n_t$ | Final, $n_{t+1}$ |
| $1.2 \times 10^3$ | $6.0 \times 10^2$ |
| $2.4 \times 10^3$ | $1.2 \times 10^3$ |
| $1.6 \times 10^3$ | $8.0 \times 10^2$ |
| $2.0 \times 10^3$ | $1.0 \times 10^3$ |
| $1.4 \times 10^3$ | $7.0 \times 10^2$ |
| $8.0 \times 10^2$ | ?? |

**40.**

| Soybean Yield Per Acre (bushels) | |
|---|---|
| Initial, $Y_t$ | Final, $Y_{t+1}$ |
| 100 | 210 |
| 50 | 110 |
| 200 | 410 |
| 75 | 160 |
| 95 | 200 |
| 250 | ?? |

**41–44** ▪ Recall the data used for Section 1.2, Exercises 49–52.

| Age, $a$ (days) | Length, $L$ (cm) | Tail Length, $T$ (cm) | Mass, $M$ (g) |
|---|---|---|---|
| 0.5 | 1.5 | 1.0 | 1.5 |
| 1.0 | 3.0 | 0.9 | 3.0 |
| 1.5 | 4.5 | 0.8 | 6.0 |
| 2.0 | 6.0 | 0.7 | 12.0 |
| 2.5 | 7.5 | 0.6 | 24.0 |
| 3.0 | 9.0 | 0.5 | 48.0 |

These data define several discrete-time dynamical systems. For example, between the first measurement (on day 0.5) and the second (on day 1.0), the length increases by 1.5 cm. Between the second measurement (on day 1.0) and the third (on day 1.5), the length again increases by 1.5 cm.

**41.** Graph the length at the second measurement as a function of length at the first, the length at the third measurement as a function of length at the second, and so on. Find the discrete-time dynamical system that reproduces the results.

**42.** Find and graph the discrete-time dynamical system for tail length.

**43.** Find and graph the discrete-time dynamical system for mass.

**44.** Find and graph the discrete-time dynamical system for age.

**45–48** ▪ Suppose students are permitted to take a test again and again until they get a perfect score of 100. We wish to write a discrete-time dynamical system describing these dynamics.

**45.** In words, what is the argument of the updating function? What is the value?

**46.** What are the domain and range of the updating function? What value do you expect if the argument is 100?

**47.** Sketch a possible graph of the updating function.

**48.** On the basis of your graph, how would a student do on her second try if she scored 20 on her first try?

**49–50** ▪ Consider the discrete-time dynamical system $b_{t+1} = 2.0b_t$ for a bacterial population (Example 1.5.1).

**49.** Write a discrete-time dynamical system for the total volume of bacteria (suppose each bacterium takes up $10^4$ $\mu m^3$).

**50.** Write a discrete-time dynamical system for the total area taken up by the bacteria (suppose the thickness is 20 $\mu$m).

**51–52** ▪ Recall the equation $h_{t+1} = h_t + 1.0$ for tree height.

**51.** Write a discrete-time dynamical system for the total volume of the cylindrical trees in Section 1.3, Exercise 27.

**52.** Write a discrete-time dynamical system for the total volume of a spherical tree (this is kind of tricky).

**53–54** ▪ Consider the following data describing the levels of a medication in the blood of two patients over the course of several days (measured in milligrams per liter).

| Day | Medication level in patient 1 (mg/L) | Medication level in patient 2 (mg/L) |
|---|---|---|
| 0 | 20.0 | 0.0 |
| 1 | 16.0 | 2.0 |
| 2 | 13.0 | 3.2 |
| 3 | 10.75 | 3.92 |

**53.** Graph three points on the updating function for the first patient. Find a linear discrete-time dynamical system for the first patient.

**54.** Graph three points on the updating function for the second patient, and find a linear discrete-time dynamical system.

**55–56** ▪ For the following discrete-time dynamical systems, compute solutions starting from each of the given initial conditions. Then find the difference between the solutions as a function of time, and the ratio of the solutions as a function of time. In which cases is the difference constant, and in which cases is the ratio constant? Can you explain why?

**55.** Two bacterial populations follow the discrete-time dynamical system $b_{t+1} = 2.0b_t$, but the first starts with initial condition $b_0 = 1.0 \times 10^6$ and the second starts with initial condition $b_0 = 3.0 \times 10^5$ (in millions of bacteria).

**56.** Two trees follow the discrete-time dynamical system $h_{t+1} = h_t + 1.0$, but the first starts with initial condition $h_0 = 10.0$ m and the second starts with initial condition $h_0 = 2.0$ m.

**57–60** ▪ Derive and analyze discrete-time dynamical systems that describe the following contrasting situations.

**57.** A population of bacteria doubles every hour, but $1.0 \times 10^6$ individuals are removed after reproduction to be converted into valuable biological by-products. The population begins with $b_0 = 3.0 \times 10^6$ bacteria.

a. Find the population after 1, 2, and 3 hours.

b. How many bacteria were harvested?

c. Write the discrete-time dynamical system.

d. Suppose you waited to harvest bacteria until the end of 3 hours. How many could you remove and still match the population $b_3$ found in part **a**? Where did all the extra bacteria come from?

58. Suppose that a population of bacteria doubles every hour but that $1.0 \times 10^6$ individuals are removed **before** reproduction to be converted into valuable biological by-products. Suppose the population begins with $b_0 = 3.0 \times 10^6$ bacteria.
   a. Find the population after 1, 2, and 3 hours.
   b. Write the discrete-time dynamical system.
   c. How does the population compare with that in the previous problem? Why is it doing worse?

59. Suppose the fraction of individuals with some superior gene increases by 10% each generation.
   a. Write the discrete-time dynamical system for the fraction of organisms with the gene (denote the fraction at time $t$ by $f_t$ and figure out the formula for $f_{t+1}$).
   b. Write the solution, starting from an initial condition of $f_0 = 0.0001$.
   c. Will the fraction reach 1.0? Does the discrete-time dynamical system make sense for all values of $f_t$?

60. The Weber-Fechner law describes how human beings perceive differences. Suppose, for example, that a person first hears a tone with a frequency of 400 hertz (cycles per second). He is then tested with higher tones until he can hear the difference. The ratio between these values describes how well this person can hear differences.
   a. Suppose the next tone he can distinguish has a frequency of 404 hertz. What is the ratio?
   b. According to the Weber-Fechner law, the next higher tone will be greater than 404 by the same ratio. Find this tone.
   c. Write the discrete-time dynamical system for this person. Find the fifth tone he can distinguish.
   d. Suppose the experiment is repeated on a musician, and she manages to distinguish 400.5 hertz from 400 hertz. What is the fifth tone she can distinguish?

**61–62** ▪ The total mass of a population of bacteria will change if the number of bacteria changes, if the mass per bacterium changes, or if both of these variables change. Try to derive a discrete-time dynamical system for the total mass in the following situations.

61. The number of bacteria doubles each hour, and the mass of each bacterium triples during the same time.

62. The number of bacteria doubles each hour, and the mass of each bacterium increases by $1.0 \times 10^{-9}$ g. What seems to go wrong with this calculation? Can you explain why?

## 1.6 Analysis of Discrete-Time Dynamical Systems

We have defined discrete-time dynamical systems that describe what happens during a single time step and have defined the solution as the sequence of values taken on over many time steps. Often enough, finding a formula for the solution is difficult or impossible. Nonetheless, we can often deduce the behavior of the solution with simpler methods. This section introduces two such methods. **Cobwebbing** is a graphical technique that makes it possible to sketch solutions without computing anything. Algebraically, we will learn how to solve for **equilibria,** points where the discrete-time dynamical system leaves the value unchanged.

### Cobwebbing: A Graphical Solution Technique

Suppose we have a general discrete-time dynamical system

$$m_{t+1} = f(m_t)$$

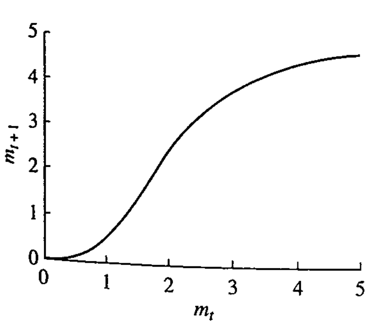



**FIGURE 1.6.83**
Graph of the updating function

with updating function graphed in Figure 1.6.83. By adding the diagonal (the line $m_{t+1} = m_t$) to the graph, we can find the behavior of solutions graphically. The technique is called **cobwebbing.**

Suppose we are given some initial condition $m_0$. To find $m_1$, we must remember the meaning of the updating function,

$$m_1 = f(m_0)$$

Graphically, $m_1$ is the coordinate of the vertical point on the graph of the updating function directly above $m_0$ (Figure 1.6.84a). Similarly, $m_2$ is the coordinate of the vertical point on the graph of the updating function directly above $m_1$, and so on.

The missing step is moving $m_1$ from the vertical axis onto the horizontal axis. The trick is to **reflect** it off the diagonal line that has equation $m_{t+1} = m_t$. Move the point $(m_0, m_1)$ horizontally until it intersects the diagonal. Moving a point horizontally does

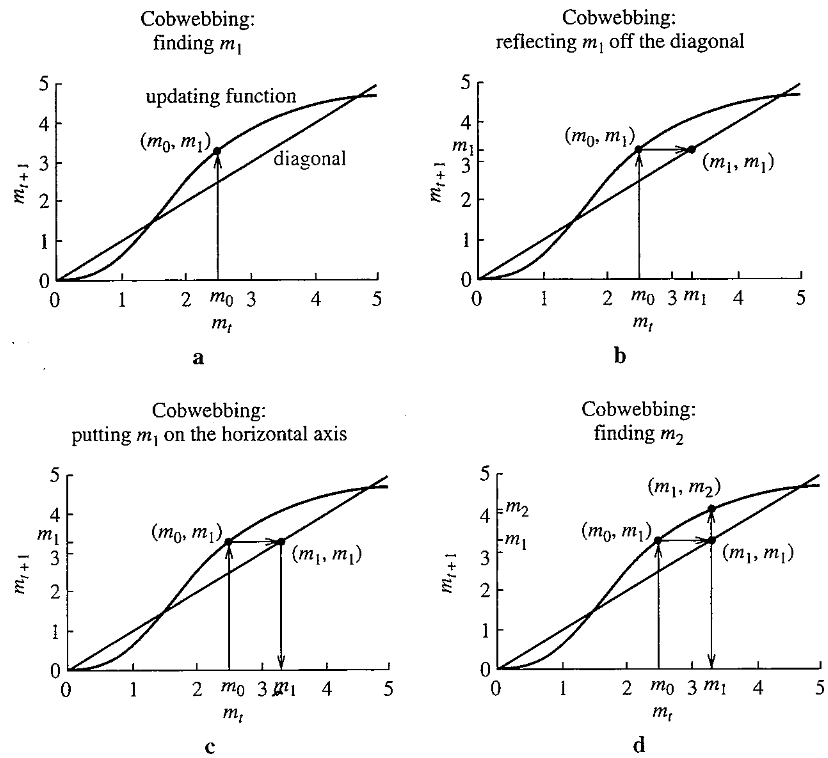



**FIGURE 1.6.84**
Cobwebbing: The first steps

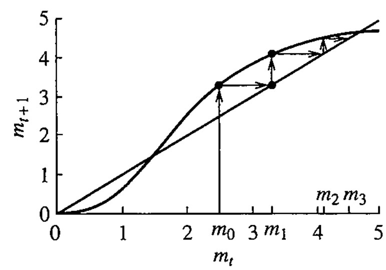



**FIGURE 1.6.85**
Cobwebbing

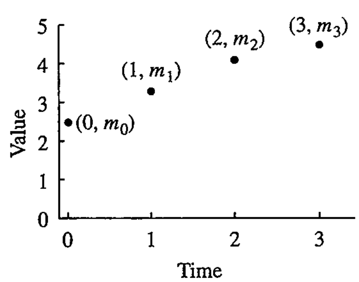



**FIGURE 1.6.86**
The solution derived from a cobweb diagram

not change the vertical coordinate. The intersection with the diagonal occurs at th point $(m_1, m_1)$ (Figure 1.6.84b). The point $(m_1, 0)$ lies directly below (Figure 1.6.84c

What have we done? Starting from the initial value $m_0$, plotted on the horizont axis, we used the updating function to find $m_1$ on the vertical axis and the reflectin trick to project $m_1$ onto the horizontal axis. We then can find $m_2$ by moving vertically t the graph of the updating function (Figure 1.6.84d). To find $m_3$, we move horizontall to the diagonal to reach the point $(m_2, m_2)$, and then vertically to the point $(m_2, m_3$ Because the lines reaching all the way to the horizontal axis are unnecessary, they ar generally omitted to make the diagram more readable (Figure 1.6.85).

Having found $m_1$, $m_2$, and $m_3$ on our cobwebbing graph, we can sketch a grap of the solution that shows the measurement as a function of time. In Figure 1.6.84 we began at $m_0 = 2.5$. This is plotted as the point $(0, m_0) = (0, 2.5)$ in the solutio (Figure 1.6.86). The value $m_1$ is approximately 3.2 and is plotted as the point $(1, m_1$ in the solution. The values of $m_2$ and $m_3$ increase more slowly and are plotted thus o the graph. Without plugging numbers into the formula, we have used the graph of th updating function to figure out the behavior of a solution starting from a given initia condition.

Similarly, we can find how the concentration would behave over time if we starte from the different initial condition $m_0 = 1.2$ (Figure 1.6.87). In this case, the diagona lies below the graph of the updating function, so reflecting off the diagonal move points to the left. Therefore, the solution decreases.

The steps for cobwebbing are summarized in the following algorithm.

## ▶▶ Algorithm 1.4 Using Cobwebbing to Find a Solution

1. Graph the updating function and the diagonal.
2. Starting from the initial condition on the horizontal axis, go "up to the updatin function and over to the diagonal."

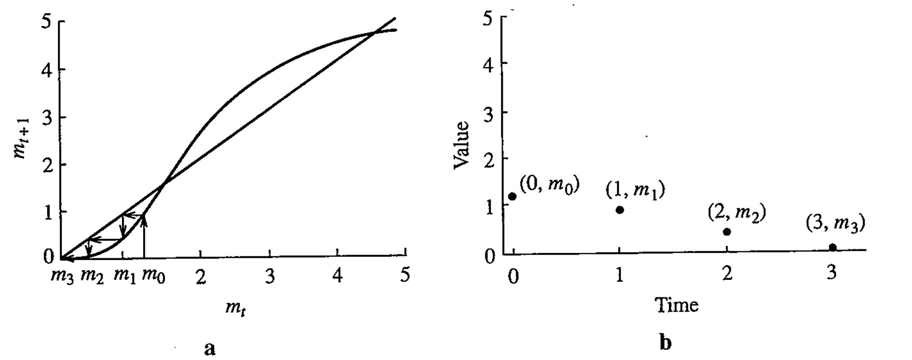



**FIGURE 1.6.87**
Cobweb and solution with a different initial condition

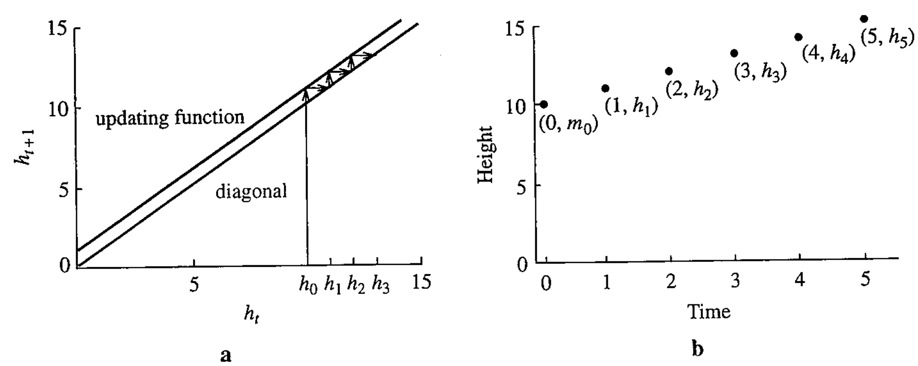



**FIGURE 1.6.88**
Cobweb and solution of tree growth model

3. Repeat going "up or down to the updating function and over to the diagonal" for as many steps as needed to find the pattern.
4. Sketch the solution at times 0, 1, 2, and so forth as the consecutive horizontal coordinates of intersections with the diagonal.

**Example 1.6.1** Cobwebbing and Solution of the Tree Growth Model

Consider the discrete-time dynamical system for a growing tree (Example 1.5.2)

$$h_{t+1} = h_t + 1.0$$

The updating function $g(h_t) = h_t + 1.0$ is a line with slope 1 and intercept 1.0, and thus it is parallel to the diagonal $h_{t+1} = h_t$ (Figure 1.6.88). Starting from an initial condition of 10.0, the cobweb moves up steadily, as does the solution (Figure 1.6.89). The graphical solution is consistent with the exact solution $h_t = 10.0 + t$ (Example 1.5.13), although it does not provide exact **quantitative** predictions.

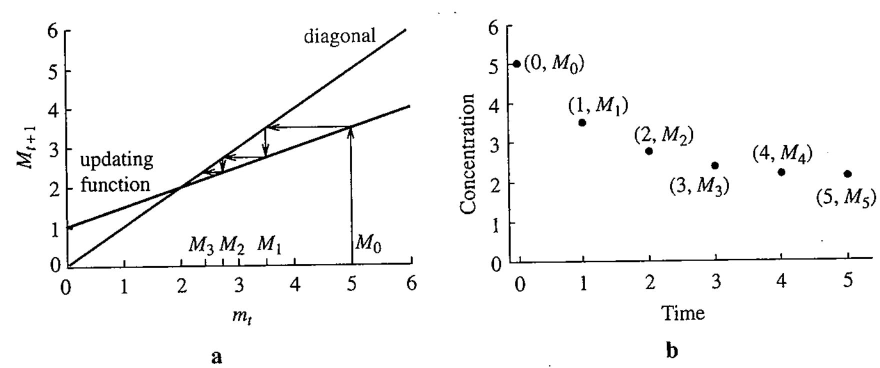



**FIGURE 1.6.89**
Cobweb and solution of the medication model: $M_0 = 5.0$

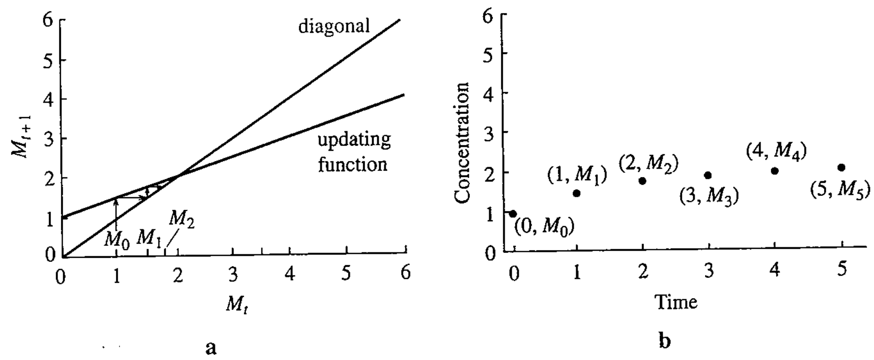



**FIGURE 1.6.90**
Cobweb and solution of the medication model: $M_0 = 1.0$

**Example 1.6.2** Cobwebbing and Solution of the Medication Model

Consider the discrete-time dynamical system for medication (Example 1.5.4)

$$M_{t+1} = 0.5M_t + 1.0$$

The updating function is a line with slope 0.5 and intercept 1, and thus it is less steep than the diagonal $M_{t+1} = M_t$. If we begin at $M_0 = 5$, the cobweb and solution decrease more and more slowly over time (Figure 1.6.89). If we begin instead at $M_0 = 1$, the cobweb and solution increase over time (Figure 1.6.90).

## Equilibria: Graphical Approach

The points where the graph of the updating function intersects the diagonal play a special role in cobweb diagrams. These points also play an essential role in understanding the behavior of discrete-time dynamical systems. Consider the discrete-time dynamical systems plotted in Figure 1.6.91. The first describes a population of plants (denoted by $P_t$ at time $t$) and the second a population of birds (denoted by $B_t$ at time $t$). Each graph includes the diagonal line used in cobwebbing.

If we begin cobwebbing from an initial condition where the graph of the updating function lies *above* the diagonal, the population increases (Figure 1.6.92a). In contrast, if we begin cobwebbing from an initial condition where the graph of the updating function lies *below* the diagonal, the population decreases (Figure 1.6.92b). The plant population will thus increase if the initial condition lies below the crossing point, but it will decrease if it lies above.

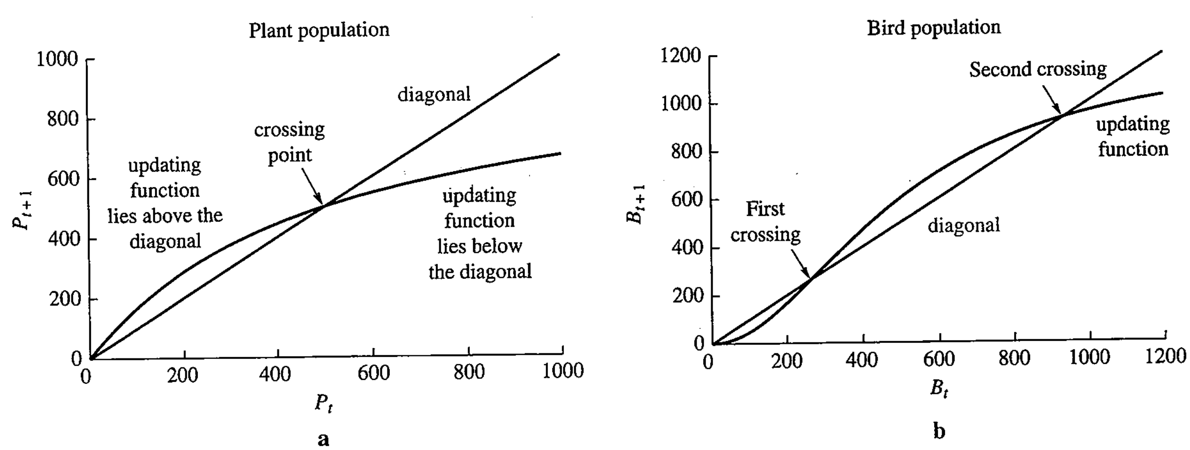



**FIGURE 1.6.91**
Dynamics of two populations

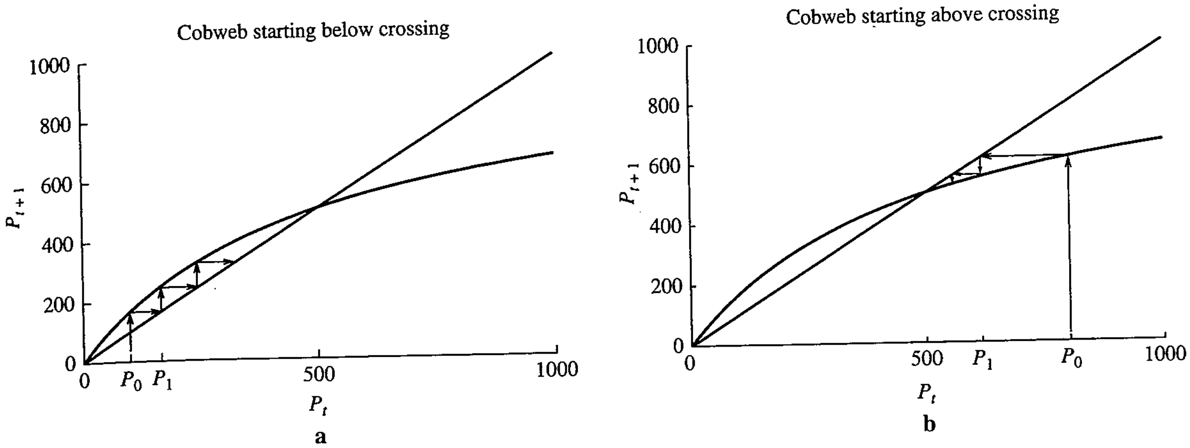



**FIGURE 1.6.92**
Behavior of plant population starting from two initial conditions

Similarly, the updating function for the bird population lies below the diagonal for initial conditions less than the first crossing, and the population decreases (Figure 1.6.93a). The updating function is above the diagonal for initial conditions between the crossings, and the population increases (Figure 1.6.93b). Finally, the updating function is again below the diagonal for initial conditions greater than the second crossing, and the population decreases (Figure 1.6.93c).

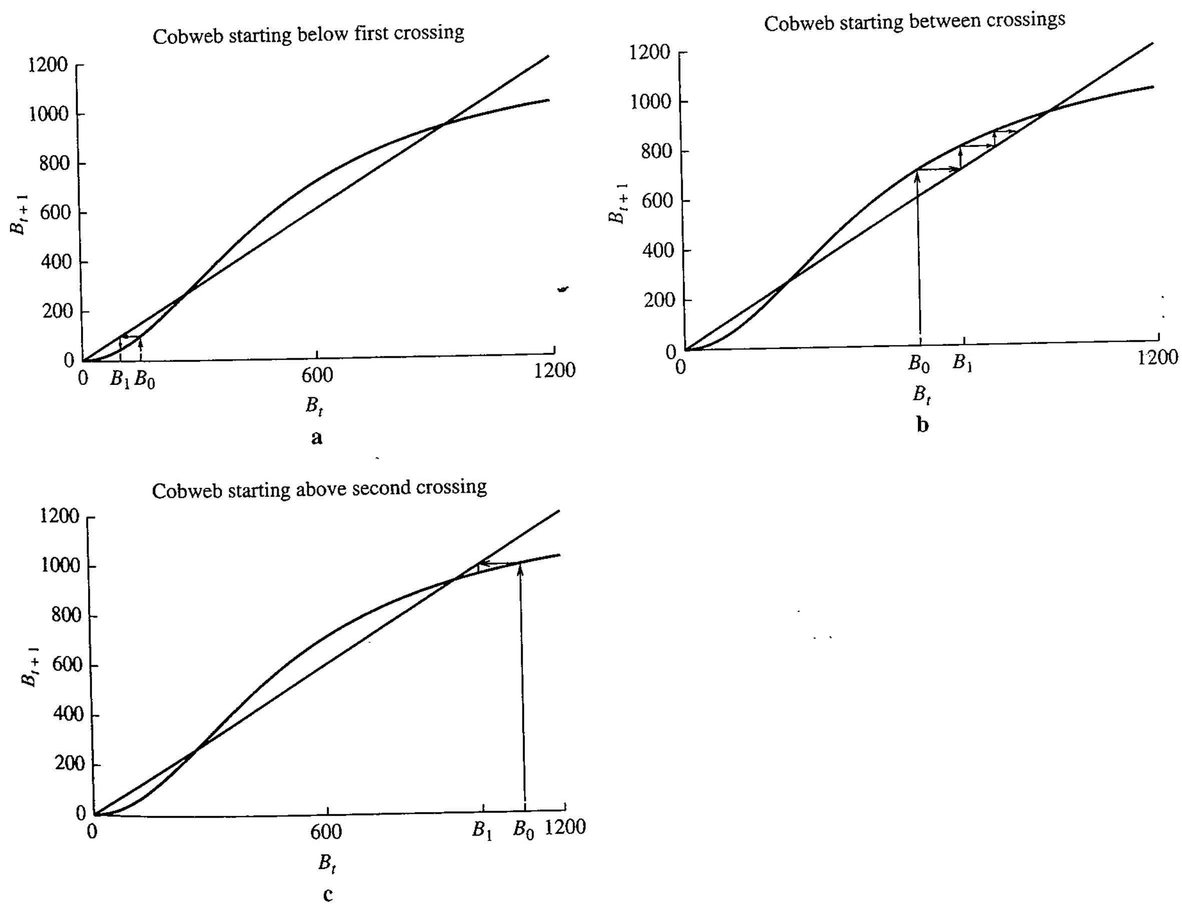



**FIGURE 1.6.93**
Behavior of bird population starting from three initial conditions

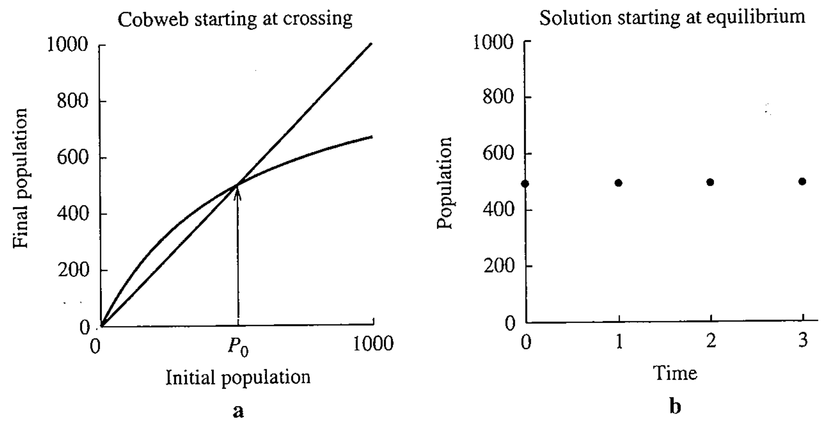



FIGURE 1.6.94
Behavior of plant population starting from an equilibrium

What happens where the updating function crosses the diagonal? At these points, the population neither increases nor decreases and thus remains the same. Such a point is called an **equilibrium.**

**Definition 1.11** A point $m^*$ is called an equilibrium of the discrete-time dynamical system

$$m_{t+1} = f(m_t)$$

if $f(m^*) = m^*$.

This definition says that the discrete-time dynamical system leaves $m^*$ unchanged. These points can be found graphically by looking for intersections of the graph of the updating function with the diagonal line.

When there is more than one equilibrium, they are called **equilibria**. The plant population has two equilibria, one at $P = 0$ and one at $P = 500$. If we start cobwebbing from an initial condition exactly equal to an equilibrium, not much happens. The cobweb goes up to the crossing point and gets stuck there (Figure 1.6.94a). The solution is a horizontal sequence of dots (Figure 1.6.94b).

Why does the graphical method for finding equilibria work? The diagonal has equation

$$m_{t+1} = m_t$$

and can be thought of as a discrete-time dynamical system that leaves *all* inputs unchanged and always returns an output equal to its input. The intersections of the graph of the updating function with the diagonal are thus points where the updating function leaves its input unchanged. These are the equilibria.

## Equilibria: Algebraic Approach

When we know the formula for the discrete-time dynamical system, we can sometimes solve for the equilibria algebraically.

**Example 1.6.3** The Equilibrium of the Medication Discrete-time Dynamical System

Recall the discrete-time dynamical system for medication

$$M_{t+1} = 0.5M_t + 1.0$$

(Example 1.5.4 and Figure 1.6.95). Let $M^*$ stand for an equilibrium. The equation for equilibrium says that $M^*$ is unchanged by the discrete-time dynamical system, or

$$M^* = 0.5M^* + 1.0$$

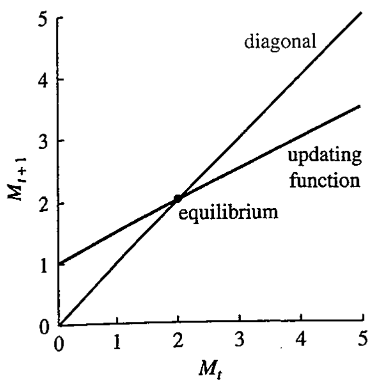



FIGURE 1.6.95
Equilibrium of the medication discrete-time dynamical system

We can solve this linear equation.

$$
\begin{aligned}
M^* &= 0.5M^* + 1.0 && \text{the original equation} \\
M^* - 0.5M^* &= 1.0 && \text{subtract } 0.5M^* \text{ to get unknowns on one side} \\
0.5M^* &= 1.0 && \text{do the subtraction} \\
M^* &= \frac{1.0}{0.5} = 2.0 && \text{divide by 0.5}
\end{aligned}
$$

The equilibrium value is 2.0 mg/l. We can check this by plugging $M_t = 2.0$ into the discrete-time dynamical system, finding that

$$M_{t+1} = 0.5 \cdot 2.0 + 1.0 = 2.0$$

A concentration of 2.0 is indeed unchanged over a course of days. Furthermore, we have seen that solutions tend to approach the equilibrium (Examples 1.5.14 and 1.5.15).

**Example 1.6.4** The Equilibrium of the Bacterial Discrete-time Dynamical System

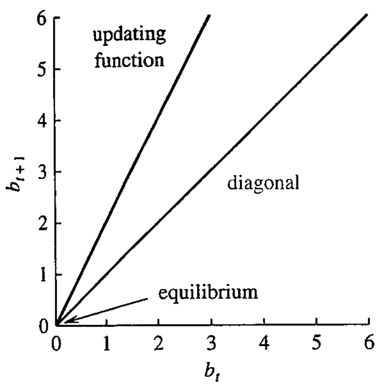



FIGURE 1.6.96
Equilibrium of the bacterial discrete-time dynamical system

To find the equilibria for the bacterial population discrete-time dynamical system

$$b_{t+1} = 2b_t$$

(Example 1.5.1 and Figure 1.6.96), we write the equation for equilibria,

$$b^* = 2b^*$$

We then solve this equation

$$
\begin{aligned}
b^* &= 2b^* && \text{the original equation} \\
b^* - b^* &= 2b^* - b^* && \text{subtract } b^* \text{ from both sides} \\
0 &= b^* && \text{do the subtraction}
\end{aligned}
$$

Consistent with our picture, the only equilibrium is at $b_t = 0$. The only number that remains the same after doubling is 0.

**Example 1.6.5** A Discrete-time Dynamical System with No Equilibrium

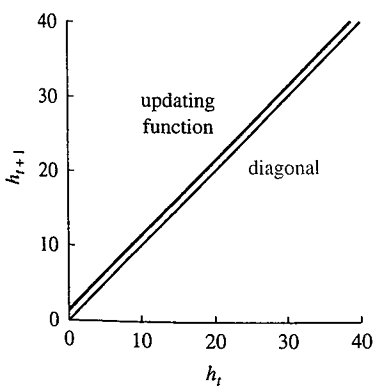



FIGURE 1.6.97
A discrete-time dynamical system for mites

The updating function for a growing tree (Example 1.5.2) following the discrete-time dynamical system

$$h_{t+1} = h_t + 1.0$$

has a graph that is parallel to the diagonal Figure 1.6.97. To solve for the equilibria, we try

$$
\begin{aligned}
h^* &= h^* + 1 && \text{the equation for the equilibrium} \\
h^* - h^* &= 1 && \text{subtract } h^* \text{ to get unknowns on one side} \\
0 &= 1 && \text{do the subtraction}
\end{aligned}
$$

This looks bad. The graph of the updating function and the graph of the diagonal do not intersect because they are parallel lines. Something that grows 1.0 m per year cannot remain unchanged.

**Example 1.6.6** A Biologically Unrealistic Equilibrium

The graph of the updating function associated with a mite population (Example 1.5.3) that follows the discrete-time dynamical system

$$x_{t+1} = 2x_t + 30$$

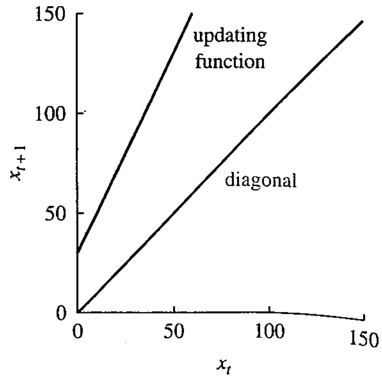



**FIGURE 1.6.98**
The discrete-time dynamical system for mites

lies above the diagonal for all values of $x_t$ (Figure 1.6.98). To solve for the equilibria, try

| | |
|---|---|
| $x^* = 2x^* + 30$ | the equation for the equilibrium |
| $x^* - 2x^* = 30$ | subtract $2x^*$ to get unknowns on one side |
| $-x^* = 30$ | do the subtraction |
| $x^* = -30$ | divide both sides by $-1$ |

This looks like nonsense. However, if we check by substituting $x_t = -30$ into the discrete-time dynamical system, we find

$$x_{t+1} = 2 \cdot (-30) + 30 = -30$$

which is indeed equal to $x_t$.

Although there is a **mathematical** equilibrium, there is no **biological** equilibrium. If we extend the graph to include biologically meaningless negative values, we see that the graph of the updating function does intersect the diagonal (Figure 1.6.99).

**Algebra Involving Parameters** Studying the general form of a discrete-time dynamical system, using parameters instead of numbers, can sometimes simplify the algebra and make the results easier to understand. When we work with parameters, however, we must be more careful with the algebra.

## ▶▶ Algorithm 1.5 Solving for Equilibria

1. Write the equation for the equilibrium.
2. Use subtraction to move all the terms to one side, leaving 0 on the other.
3. Factor (if possible).
4. Set each factor equal to 0 and solve for the equilibria (if possible).
5. Think about the results.

As always, we begin by setting up the problem. The next three steps give a safe method to do the algebra (although the algebra may be impossible). The final step is perhaps the most important. A result is worthwhile only if it makes sense.

## Example 1.6.7 Finding Equilibria of the Bacterial Model in General

Consider the bacterial discrete-time dynamical system where the factor of 2.0 has been replaced with a general **per capita production** of $r$,

$$b_{t+1} = rb_t$$

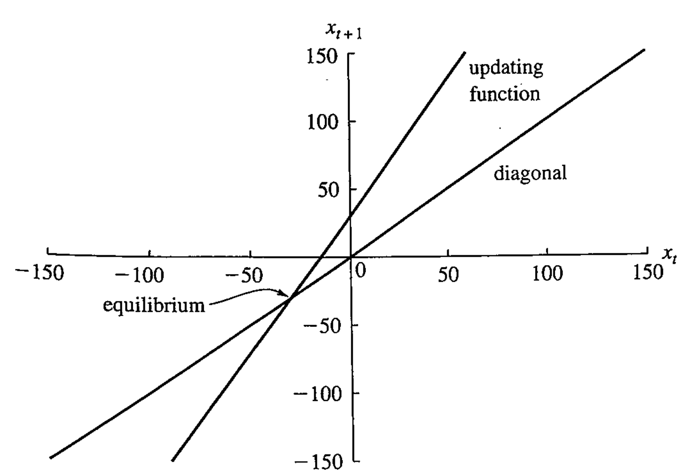



**FIGURE 1.6.99**
Extending the discrete-time dynamical system for mites to include a negative domain

We will study this form in more detail in Section 1.7. The factor $r$ describes how much the population grows (or declines) in 1 hour. Applying Algorithm 1.5 gives

$$\begin{aligned} b^* &= rb^* && \text{the equation for the equilibrium} \\ b^* - rb^* &= 0 && \text{move everything to one side} \\ b^*(1-r) &= 0 && \text{factor out the common factor of } b^* \\ b^* = 0 &\text{ or } 1 - r = 0 && \text{set both factors to 0} \\ b^* = 0 &\text{ or } r = 1 && \text{solve each equation} \end{aligned}$$

There are two possibilities. The first matches what we found earlier; a population of 0 is at equilibrium. This makes sense because an extinct population remains extinct. The second is new. If the per capita production $r$ is exactly 1, every value of $b_t$ is an equilibrium. In this case, each bacterium exactly replaces itself. The population size will remain the same no matter what its size, even though the individual bacteria are reproducing and dying.

**Example 1.6.8** Equilibria of the Medication Model with a Dosage Parameter

Consider the medication discrete-time dynamical system with the parameter $S$,

$$M_{t+1} = 0.5M_t + S$$

where $S$ represents the daily dosage. The algorithm for finding equilibria gives

$$\begin{aligned} M^* &= 0.5M^* + S && \text{the equation for the equilibrium} \\ M^* - 0.5M^* - S &= 0 && \text{move everything to one side} \\ 0.5M^* - S &= 0 && \text{simplify} \\ M^* &= 2.0S && \text{nothing to factor, solve for } M^* \end{aligned}$$

The equilibrium value is proportional to $S$, the daily dosage.

**Example 1.6.9** Equilibria of the Medication Model with Absorption

Consider the medication discrete-time dynamical system with parameter $\alpha$,

$$M_{t+1} = (1-\alpha)M_t + 1.0$$

where the parameter $\alpha$ represents the fraction of existing medication absorbed by the body during a given day. For example, if $\alpha = 0.1$, 10% of the medication is absorbed by the body and 90% remains.

$$\begin{aligned} M^* &= (1-\alpha)M^* + 1.0 && \text{the equation for the equilibrium} \\ M^* - (1-\alpha)M^* - 1.0 &= 0 && \text{move everything to one side} \\ M^* - M^* + \alpha M^* - 1.0 &= 0 && \text{distribute negative sign through quantity} \\ \alpha M^* - 1.0 &= 0 && \text{cancel } M^* - M^* \\ M^* &= \frac{1.0}{\alpha} && \text{solve for } M^* \end{aligned}$$

The equilibrium value is proportional to the reciprocal of $\alpha$ and thus is smaller when the fraction absorbed is larger. If $\alpha = 0.1$, the equilibrium is

$$M^* = \frac{1.0}{0.1} = 10.0$$

In contrast, if the body absorbs 50% of the medication each day, leading to a larger value of $\alpha = 0.5$, then

$$M^* = \frac{1.0}{0.5} = 2.0$$

The body that absorbs more reaches a lower equilibrium.

**Example 1.6.10** Equilibria of the Medication Model with Two Parameters

Consider the medication discrete-time dynamical system with both parameters from Examples 1.6.8 and 1.6.9,

$$M_{t+1} = (1-\alpha)M_t + S$$

The algorithm for finding equilibria gives

| | |
|---|---|
| $M^* = (1-\alpha)M^* + S$ | the equation for the equilibrium |
| $M^* - (1-\alpha)M^* - S = 0$ | move everything to one side |
| $M^* - M^* + \alpha M^* - S = 0$ | distribute negative sign through quantity |
| $\alpha M^* - S = 0$ | cancel $M^* - M^*$ |
| $M^* = \dfrac{S}{\alpha}$ | solve for $M^*$ |

The equilibrium value is larger if $S$ is larger or if $\alpha$ is smaller. This makes sense because the equilibrium concentration can be increased in two ways: by increasing the dosage or by decreasing the fraction absorbed.

**Summary** We have developed a graphical technique called cobwebbing to estimate solutions. By examining the diagrams used for cobwebbing, we found that intersections of the graph of the updating function with the diagonal line play a special role. These **equilibria** are points that are unchanged by the discrete-time dynamical system. Algebraically, we find equilibria by solving the equation that describes such points. With a little extra care, we can often solve for equilibria in general, without substituting numerical values for the parameters. Solving the equations in this way can help clarify the underlying biological process.

## 1.6 Exercises

### Mathematical Techniques

**1–2** ▪ The following steps are used to build a cobweb diagram. Follow them for the given discrete-time dynamical systems based on bacterial populations.

a. Graph the updating function.

b. Use your graph of the updating function to find the point $(b_0, b_1)$.

c. Reflect it off the diagonal to find the point $(b_1, b_1)$.

d. Use the graph of the updating function to find $(b_1, b_2)$.

e. Reflect off the diagonal to find the point $(b_2, b_2)$.

f. Use the graph of the updating function to find $(b_2, b_3)$.

g. Sketch the solution as a function of time.

**1.** The discrete-time dynamical system $b_{t+1} = 2.0b_t$ with $b_0 = 1.0$.

**2.** The discrete-time dynamical system $n_{t+1} = 0.5n_t$ with $n_0 = 1.0$.

**3–6** ▪ Cobweb the following discrete-time dynamical systems for three steps, starting from the given initial condition. Compare with the solution found earlier.

**3.** $v_{t+1} = 1.5v_t$, starting from $v_0 = 1220\ \mu m^3$ (as in Section 1.5, Exercise 5).

**4.** $l_{t+1} = l_t - 1.7$, starting from $l_0 = 13.1$ cm (as in Section 1.5, Exercise 6).

**5.** $n_{t+1} = 0.5n_t$, starting from $n_0 = 1200$ (as in Section 1.5, Exercise 7).

**6.** $M_{t+1} = 0.75M_t + 2.0$ starting from $M_0 = 16.0$ mg/l (as in Section 1.5, Exercise 8).

**7–12** ▪ Graph the updating functions associated with the following discrete-time dynamical systems, and cobweb for five steps, starting from the given initial condition.

**7.** $x_{t+1} = 2x_t - 1$, starting from $x_0 = 2$.

**8.** $z_{t+1} = 0.9z_t + 1$, starting from $z_0 = 3$.

**9.** $w_{t+1} = -0.5w_t + 3$, starting from $w_0 = 0$.

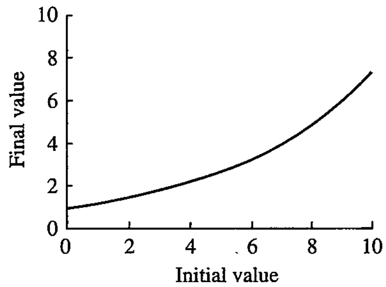
**10.** $x_{t+1} = 4 - x_t$, starting from $x_0 = 1$ (as in Section 1.5, Exercise 25).

**11.** $x_{t+1} = \dfrac{x_t}{1 + x_t}$, starting from $x_0 = 1$ (as in Section 1.5, Exercise 23).

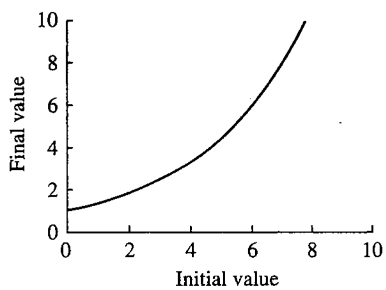
**12.** $x_{t+1} = \dfrac{x_t}{x_t - 1}$, starting from $x_0 = 3$ (as in Section 1.5, Exercise 26). Graph for $x_t > 1$.

**13–16** ▪ Find the equilibria of the following discrete-time dynamical system from the graphs of their updating functions. Label the coordinates of the equilibria.

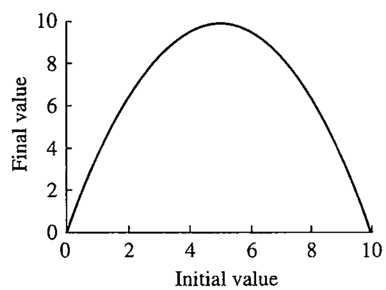
**13.**



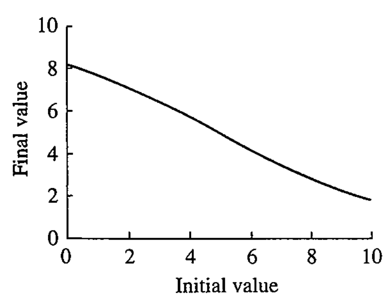

**14.**




**15.**




**16.**




**17–18** ▪ Sketch graphs of the following updating functions over the given range, and mark the equilibria. Find the equilibria algebraically if possible.

**17.** $f(x) = x^2$ for $0 \le x \le 2$

**18.** $g(y) = y^2 - 1$ for $0 \le y \le 2$

**19–22** ▪ Graph the following discrete-time dynamical systems. Solve for the equilibria algebraically, and identify equilibria and the regions where the updating function lies above the diagonal on your graph.

**19.** $c_{t+1} = 0.5c_t + 8.0$, for $0 \le c_t \le 30$

**20.** $b_{t+1} = 3b_t$, for $0 \le b_t \le 10$

**21.** $b_{t+1} = 0.3b_t$, for $0 \le b_t \le 10$

**22.** $b_{t+1} = 2.0b_t - 5.0$, for $0 \le b_t \le 10$

**23–30** ▪ Find the equilibria of the following discrete-time dynamical systems. Compare with the results of your cobweb diagram from the earlier problem.

**23.** $v_{t+1} = 1.5v_t$ (as in Section 1.5, Exercise 5)

**24.** $l_{t+1} = l_t - 1.7$ (as in Section 1.5, Exercise 6)

**25.** $x_{t+1} = 2x_t - 1$ (as in Section 1.6, Exercise 7)

**26.** $z_{t+1} = 0.9z_t + 1$ (as in Section 1.6, Exercise 8)

**27.** $w_{t+1} = -0.5w_t + 3$ (as in Section 1.6, Exercise 9)

**28.** $x_{t+1} = 4 - x_t$ (as in Section 1.6, Exercise 10)

**29.** $x_{t+1} = \dfrac{x_t}{1 + x_t}$ (as in Section 1.6, Exercise 11)

**30.** $x_{t+1} = \dfrac{x_t}{x_t - 1}$ for $x_t > 1$ (as in Section 1.6, Exercise 12)

**31–34** ▪ Find the equilibria of the following discrete-time dynamical systems that include parameters. Identify values of the parameter for which there is no equilibrium, for which the equilibrium is negative, and for which there is more than one equilibrium.

**31.** $w_{t+1} = aw_t + 3$

**32.** $x_{t+1} = b - x_t$

**33.** $x_{t+1} = \dfrac{ax_t}{1 + x_t}$

**34.** $x_{t+1} = \dfrac{x_t}{x_t - K}$

## Applications

**35–40** ▪ Cobweb the following discrete-time dynamical systems for five steps, starting from the given initial condition.

**35.** An alternative tree growth discrete-time dynamical system with form $h_{t+1} = h_t + 5.0$ with initial condition $h_0 = 10$.

**36.** The mite population discrete-time dynamical system (Example 1.6.6) $x_{t+1} = 2x_t + 30$ with initial condition $x_0 = 0$.

**37.** The model defined in Section 1.5, Exercise 37, starting from an initial volume of 1420.

**38.** The model defined in Section 1.5, Exercise 38, starting from an initial length of 13.1.

**39.** The model defined in Section 1.5, Exercise 39, starting from an initial population of 800.

**40.** The model defined in Section 1.5, Exercise 40, starting from an initial yield of 20.