Modeling the Dynamics of Life

Calculus and Probability for Life Scientists

Second Edition

Frederick R. Adler

University of Utah

THOMSON
BROOKS/COLE

Australia • Canada • Mexico • Singapore • Spain • United Kingdom • United States

Chapter

1

Introduction to Discrete-Time Dynamical Systems

This chapter introduces the main tools needed to use mathematics to study biology: **functions** and **modeling.** Biological phenomena are described by **measurements,** a set of numerical values with units (like degrees or centimeters). Many **relations** between measurements are described by functions, which take one value as an input and return another as an output. We review the important functions used to describe biological systems: linear, trigonometric, power, logarithmic, and exponential functions.

Modeling is the art of taking a description of a biological phenomenon and converting it into mathematical form. Living things are characterized by change. One goal of modeling is to quantify these **dynamics** with an appropriate function. By using our understanding of the system and carefully following how a set of basic measurements change step by step, we will learn to derive an **updating function** that models the change in a **discrete-time dynamical system.** We will follow this process to derive models of bacterial population growth, gas exchange in the lung, and genetic change in a population of competing bacteria. We will develop a set of algebraic and graphical tools to deduce the dynamics that result from a particular discrete-time dynamical system.

Throughout this chapter, keep the following questions in mind:

- What biological process are we trying to describe?
- What biological questions do we seek to answer?
- What are the basic measurements and their units?
- What are the relationships between the basic measurements?
- What do results mean biologically?

1.1 Biology and Dynamics

Living systems, from cells to organisms to ecosystems, are characterized by change and dynamics. Living things grow, maintain themselves, and reproduce.Even remaining the same requires dynamical responses to a changing environment. Understanding the mechanisms behind these dynamics and deducing their consequences is crucial to understanding biology. This book uses a dynamical approach to address questions about biology.

This dynamical approach is necessarily mathematical because describing dynamics requires quantifying measurements. What is changing? How fast is it changing? What is it changing into?

In this book, we use the language of mathematics to describe quantitatively the working of living systems and develop the mathematical tools needed to compute how they change. From measurements describing the initial state of a system and a set of rules describing how change occurs, we will attempt to predict what will

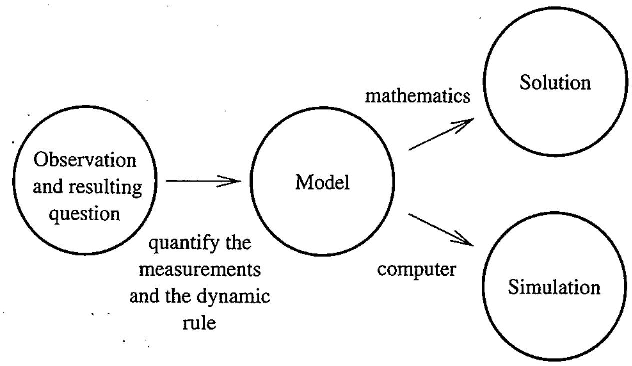


FIGURE 1.1.1

The workings of applied mathematics: the use of mathematics to answer scientific questions

happen to the system. For example, using the position and velocity of a planet (the initial state) and the laws of gravitation and inertia (the set of rules), Isaac Newton invented the mathematical methods of calculus to predict the planet's position at any future time. This example illustrates the approach of **applied mathematics,** *the use of mathematics to answer scientific questions* (Figure 1.1.1). Applied mathematics begins with scientific observations and questions, perhaps about the position of a planet, which are then quantified into a **model.** When possible, mathematical methods are developed to answer the question. In other cases, computers are used to **simulate** the process and find answers in particular cases.

The steps in applied mathematics

Step	Definition
Quantify the basic measurements.	The numerical values that describe the system
Describe a **dynamical rule.**	A description of how the basic measurements change
Develop a **model.**	A mathematical translation of the observations
Find a **solution.**	Use of mathematical methods to predict behavior
Write a **simulation.**	Use of a computer to predict behavior

This book is organized around three basic biological processes: **growth, maintenance,** and **replication.** Mathematical methods have contributed significantly to the understanding of each of these three processes. After briefly describing these contributions, we will outline the different types of models and mathematics to be used in this book.

Growth: Models of Malaria

Early in this century, Sir Ronald Ross discovered that malaria is transmitted by certain types of mosquitos. Because the disease was (and remains) difficult to treat, one promising strategy for control seemed to be reduction of the number of mosquitos. Many people thought that all the mosquitos would have to be killed to eradicate the disease. Because killing every single mosquito was impossible, it was feared that malaria might be impossible to control in this way.

Ross decided to use mathematics to convince people that mosquito control could be effective. The problem can be formulated dynamically as a problem in population growth. His first step was to **quantify the basic measurements**—in this case, the numbers of people and mosquitos with and without malaria. The **dynamical rule** describes how these numbers change. Ross knew that an uninfected person can become infected upon being bitten by an infected mosquito and that an uninfected mosquito can

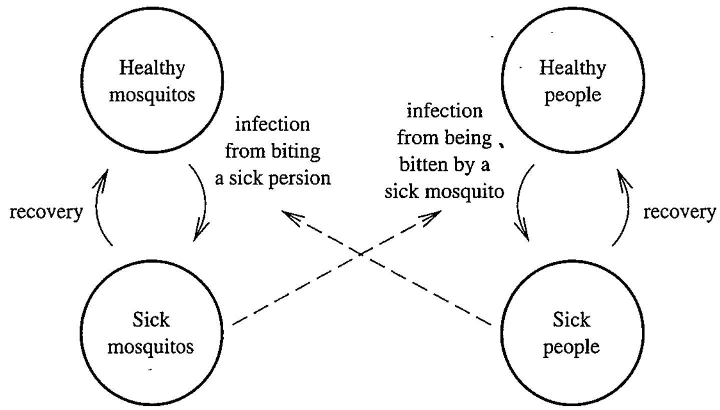


FIGURE 1.1.2
The dynamics of malaria

be infected when it bites an infected person (Figure 1.1.2). From these assumptions, he built a **mathematical model** describing the population dynamics of malaria. With this model he proved that the disease *could* be eradicated without killing every single mosquito (we will study a simple version of this model in Section 5.5). We see evidence of this today in the United States, where malaria has been virtually eliminated even though the mosquitos capable of transmitting the disease persist in many regions.

Many dynamical biological processes besides population dynamics are forms of growth. Growth in size is ubiquitous. One might use measurements of size (such as weight, height, or stomach volume) and a rule describing change in size (increase in weight due to large stomach volume) to predict the size of an organism over time. For example, one organism might add a constant amount to its weight every day, and another might add a constant fraction to its weight every day.

Organisms can also grow in complexity. For example, a tree can add branches as well as increase in size. The quantitative description of the system might include the number of branches and their ages, sizes, and pattern. The dynamical rule might give the number of new branches produced each day, the probability that a given branch divides during the next month, or the rate at which new branches are formed. From the description and rule, we could compute the number of branches as a function of age.

Maintenance: Models of Neurons

Neurons are cells that transmit information throughout the brain and body. Even the simplest neuron faces a challenging task. It must be able to amplify an appropriate incoming stimulus, transmit it to neighboring neurons, and then turn off and be ready for the next stimulus. This task is not as simple as it might seem. If we imagine the stimulus to be an input of electrical charge, a plausible sounding rule is "If electrical charge is raised above a certain level, increase it further." Such a rule works well for the first stimulus but provides no way for the cell to turn itself off. How does a neuron maintain functionality?

In the early 1950s, Hodgkin and Huxley used their own measurements of neurons to develop a mathematical model of dynamics to explain the behavior of neurons. The idea, explained in detail in Section 5.8, is that the neuron has fast and slow mechanisms to open and close specialized ion channels in response to electrical charge (Figure 1.1.3). Hodgkin and Huxley measured the dynamical behavior of these channels and showed mathematically that their mechanism explained many aspects of the functioning of neurons. They received the Nobel Prize in physiology or medicine for this work in

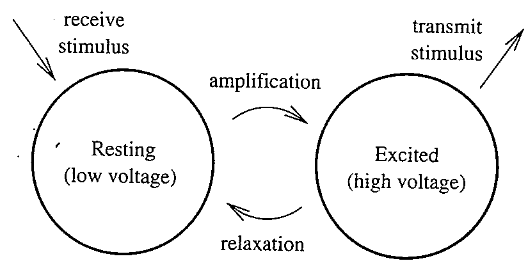


FIGURE 1.1.3
The dynamics of a neuron

1963 and, perhaps even more impressively, developed a model that is still used today to study neurons and other types of cells.

In general, maintenance of biological systems depends on preserving the distinction between inside and outside while maintaining flows of necessary materials from outside to inside, and vice versa. The neuron maintains itself at a different electrical potential from the surrounding tissue in order to be able to respond, while remaining ready to exchange ions with the outside to create the response. As applied mathematicians, we **quantify the basic measurements,** the concentrations of various substances inside and outside the cell. The **dynamical rules** express how concentrations change, generally as a function of properties of the cell membrane. Most commonly, the rule describes the process of **diffusion,** movement of materials from regions of high concentration to regions of low concentration.

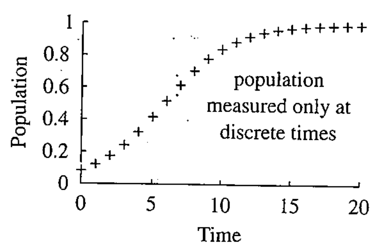


FIGURE 1.1.4
Measurements described by a discrete-time dynamical system

Replication: Models of Genetics

Although Mendel's work on genetics from the 1860s had been rediscovered around 1900, many biologists in the following decades remained unconvinced of his proposed mechanism of genetic transmission. In particular, it was unclear whether Darwin's theory of evolution by natural selection was consistent with this, or any other, proposed mechanism.

Working independently, biologists R. A. Fisher, J. B. S. Haldane, and Sewall Wright developed mathematical models of the dynamics of evolution in natural populations. These scientists **quantified the basic measurement**—in this case, the number of individuals with a particular allele (a version of a gene). Their **dynamical rules** described how many individuals in a subsequent generation would have a particular allele as a function of numerous factors, including **selection** (differential success of particular types in reproducing) and **drift** (the workings of chance). They showed that Mendel's ideas were indeed consistent with observations of evolution. This work led to the development of methods of genetic analysis used to analyze DNA sequences today. We study a simple model of selection in Section 1.10 and examine some of the consequences of Mendel's laws in Section 6.2.

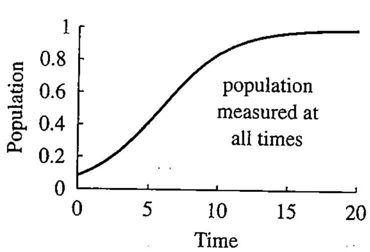


FIGURE 1.1.5
Measurements described by a continuous-time dynamical system

Types of Dynamical Systems

We will study each of the three processes—growth, maintenance, and replication—with three types of dynamical systems: discrete-time, continuous-time, and probabilistic systems. The first two types are **deterministic,** meaning that the dynamics include no chance factors. In this case, the values of the basic measurements can be predicted exactly at all future times. Probabilistic dynamical systems include chance factors, and values can be predicted only on average.

Discrete-Time Dynamical Systems

Discrete-time dynamical systems describe a sequence of measurements made at equally spaced intervals (Figure 1.1.4). These dynamical systems are described mathematically by a rule that gives the value at one time as a function of the value at the previous time.

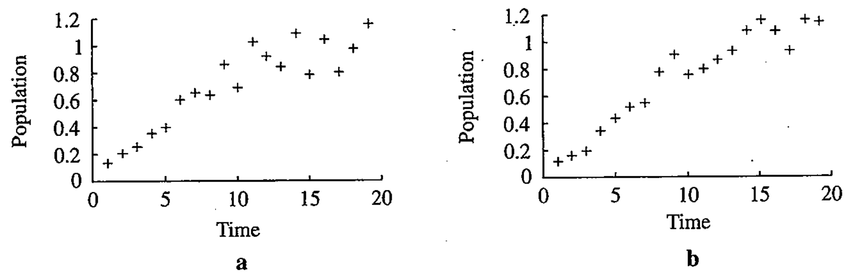


FIGURE 1.1.6
Two set of measurements described by the same probabilistic dynamical system (discrete-time case)

For example, a discrete-time dynamical system describing population growth is a rule that gives the population in one year as a function of the population in the previous year. A discrete-time dynamical system describing the concentration of oxygen in the lung is a rule that gives the concentration of oxygen in a lung after one breath as a function of the concentration after the previous breath. A discrete-time dynamical system describing the spread of a mutant allele is a rule that gives the number of mutant alleles in one generation as a function of the number in the previous generation. Mathematical analysis of the rule can make scientific predictions, such as the maximum population size, the average concentration of oxygen in the lung, or the final number of mutant alleles. The study of these systems requires the mathematical methods of modeling (Chapter 1) and **differential calculus** (Chapters 2 and 3).

Continuous-Time Dynamical Systems

Continuous-time dynamical systems, usually known as **differential equations,** describe measurements that are collected continuously (Figure 1.1.5). A differential equation consists of a rule that gives the **instantaneous rate of change** of a set of measurements. The beauty of differential equations is that information about a system at one time is sufficient to predict the state of a system at all future times. For example, a continuous-time dynamical system describing the growth of a population is a rule that gives the rate of change of population size as a function of the population size itself. The study of these systems requires the mathematical methods of **integral calculus** (Chapters 4 and 5).

Probabilistic Dynamical Systems

Probabilistic dynamical systems describe measurements, in either discrete or continuous time, that are affected by random factors. In the discrete time case, data are collected at equally spaced time intervals (Figure 1.1.6). The rule indicating how the measurements at one time depend on measurements at the previous time includes random factors. Rather than knowing the next measurements with certainty, we know only a set of possible outcomes and their associated probabilities and can therefore predict the outcome only in a probabilistic or statistical sense. For example, a probabilistic dynamical system describing population growth is a rule that gives the **probability** that a population has a particular size in one year as a function of the population in the previous year. The study of such systems requires the mathematical methods of **probability theory** (Chapters 6 and 7).

1.2 Variables, Parameters, and Functions in Biology

Quantitative science is built upon measurements. Mathematics provides the notation for describing and thinking about measurements and relations between them. In fact, the development of clear notation for measurements and relations was essential for the progress of modern science. In this section, we develop the algebraic notation needed to describe measurements, introducing **variables** to describe measurements that change *during* the course of an experiment and **parameters** that remain constant during an experiment but can change *between* different experiments. The most important types of relations between measurements are described with **functions**, where the value of one can be computed from the value of the other. We will review how to graph functions, how to combine them with **addition, multiplication,** and **composition,** and how to recognize whether a function has an **inverse** and how to compute it.

Describing Measurements with Variables, Parameters, and Graphs

Algebra uses letters or other symbols to represent numerical quantities.

Definition 1.1 A **variable** is a symbol that represents a measurement that can change during the course of an experiment.

A simple experiment measures how the population of bacteria in a culture changes over time. Because two changing quantities are being measured (time and bacterial population), we need two variables to represent them. In applied mathematics, we choose variables that remind us of the measurements they represent. In this case, we can use t to represent time and b to represent the population of bacteria. Because there are fewer letters than quantities to be measured, the same letter can be used to represent different quantities in different problems. Always define variables explicitly when writing a model, and be sure to check their definitions when reading one.

Example 1.2.1 Describing Bacterial Population Growth

t	b
0.0	1.00
1.0	1.24
2.0	1.95
3.0	3.14
4.0	4.81
5.0	6.95
6.0	9.57

The table lists measurements of bacterial population size (in millions), denoted by the variable b, at different times t after the beginning of an experiment.

Thinking about data is often easier with a graph. Graphs are drawn on the **Cartesian coordinate** plane—that is, by using two perpendicular number lines called **axes** to describe two numbers (Figure 1.2.7). The input is placed on the horizontal axis (sometimes called the x-axis) and the output is placed on the vertical axis (sometimes called the y-axis). The crossing point of the two axes is the **origin.** The axes are labeled with the variable name, the measurement it represents, and often the units of measurement (Section 1.3). **Never draw a graph without labeling the axes.**

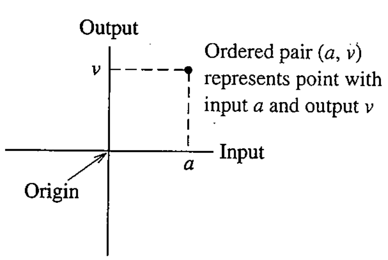


FIGURE 1.2.7
The components of a graph using Cartesian coordinates

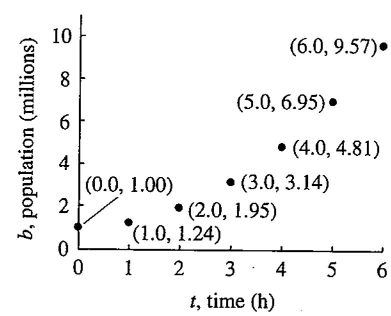


FIGURE 1.2.8
Results of bacterial growth experiment: Cartesian coordinates

Example 1.2.2 Graphing Data Describing Bacterial Population Growth

To graph the six data points in Example 1.2.1, plot each point by moving a distance t to the right of the origin along the horizontal axis and a distance b up from the origin along the vertical axis (Figure 1.2.8). For example, the data point at $t = 4.0$ is graphed by moving a distance 4.0 to the right of the origin on the horizontal axis and a distance 4.81 up from the origin on the vertical axis.

Example 1.2.3 Describing the Dynamics of a Bacterial Population

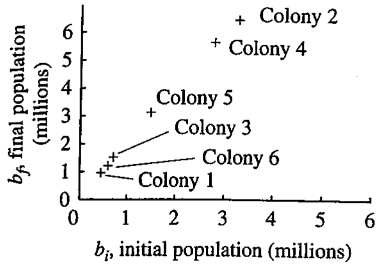


FIGURE 1.2.9

Results of alternative bacterial growth experiment

Suppose several bacterial cultures with different initial population sizes are grown in controlled conditions for one hour and then carefully counted. The population size acts as the basic measurement at both times. We must use different variables to represent these values, and we choose to use **subscripts** to distinguish them. In particular, we let b_i (for the **initial** population) represent the population at the beginning of the experiment, and we let b_f (for the **final** population) represent the population at the end. The following table and Figure 1.2.9 present the results for six colonies.

Colony	Initial Population, b_i	Final Population, b_f
1	0.47	0.94
2	3.30	6.60
3	0.73	1.46
4	2.80	5.60
5	1.50	3.00
6	0.62	1.24

Experiments of this sort form the basis of discrete-time dynamical systems (Section 1.5) and are the central topic of this chapter.

Experiments are done in a particular set of controlled conditions that remain constant during the experiment. However, these conditions might differ between experiments.

Definition 1.2 A **parameter** is a symbol that represents a measurement that does not change during the course of an experiment.

Different experiments tracking the growth of bacterial populations over time might take place at temperatures that are constant during an experiment but differ between experiments. The temperature, in this case, is represented by a parameter. Parameters, like variables, are represented by symbols that recall the measurement. We can use T to represent temperature. In applied mathematics, capital letters (like T) and small letters (like t) are often used in the same problem to represent different quantities.

Example 1.2.4 Variables and Parameters

Suppose a biologist measures growing bacterial populations at three different temperatures. During the course of each experiment, the temperature is held constant, while the population changes.

t	b when $T = 25°C$	b when $T = 35°C$	b when $T = 45°C$
0.0	1.00	1.00	1.00
1.0	1.14	1.45	0.93
2.0	1.30	2.10	0.87
3.0	1.48	3.03	0.81
4.0	1.68	4.39	0.76
5.0	1.92	6.36	0.70
6.0	2.18	9.21	0.66

Figure 1.2.10 compares the population sizes of the three populations. The population grows most quickly at the intermediate temperature of 35°C and declines at the high temperature of 45°C.

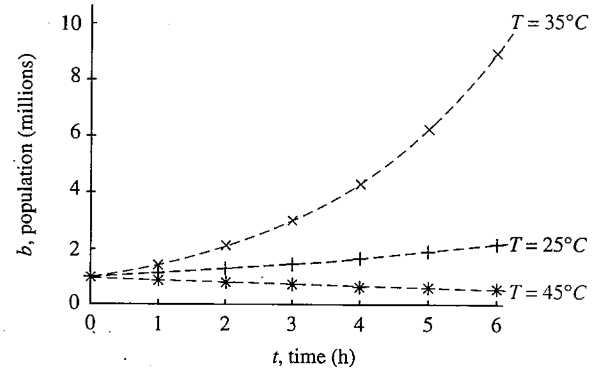


FIGURE 1.2.10
Results of bacterial growth experiment at three temperatures

Describing Relations Between Measurements with Functions

Numbers describe measurements, and **functions** describe **relations** between measurements. For example, bacterial population growth relates two measurements, denoted by the variables t and b. In general, a **relation** between two variables is the set of all pairs of values that occur.

Example 1.2.5 A Relation Between Temperature and Population Size

T	P
25.0	2.18
25.0	2.45
25.0	2.10
25.0	3.03
35.0	9.21
35.0	7.39
35.0	6.36
45.0	0.66
45.0	0.93

Suppose the temperature T and final population size P are measured for 9 populations, with the results shown in the table and Figure 1.2.11. These values could result from repeating the experiment in Example 1.2.4 several times and measuring the population at $t = 6.0$.

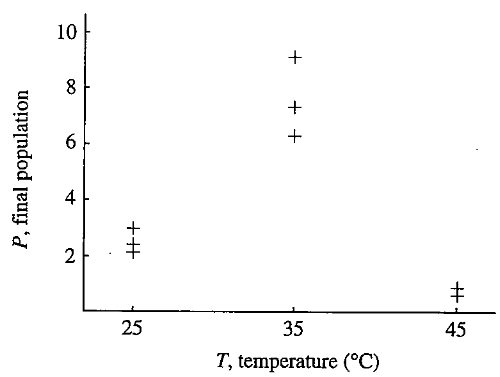


FIGURE 1.2.11
Final population size at three temperatures

Different values of the population P are related to each temperature, perhaps, because of differences in experimental conditions.

A **function** describes a specific, and important, type of relation. A function is a mathematical object that takes something (such as a number) as input, performs an operation on it, and returns a unique new object (such as another number) as output. The input is called the **argument** (or the **independent variable**) and the output is called the **value** (or the **dependent variable**) (Figure 1.2.12). The set of all possible things that a function can accept as inputs is called the **domain.** The set of all possible things that a function *can* return as outputs is called the **codomain,** and the set of all things the function *does* return as outputs is called the **range.**

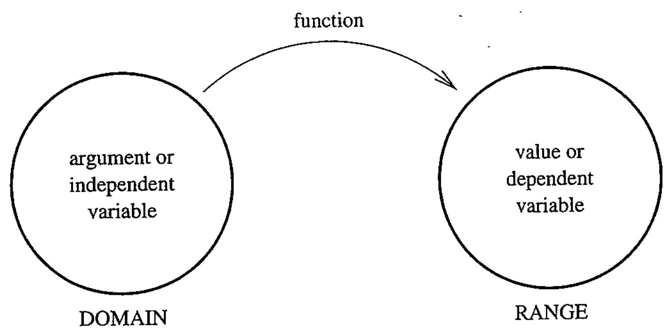


FIGURE 1.2.12
The basic terminology for describing a function

Example 1.2.6 Data That Can Be Described by a Function

The data in Example 1.2.1 can be described by a function. Each value of the input t is associated with only a single value of the output b.

Example 1.2.7 Graphing a Function from its Formula

To graph a function from a formula, it is easiest to start by plugging in some representative arguments. Suppose we wish to graph the function $f(x) = 4 + x - x^2$ for $x \geq 0$ (restricting the domain to positive numbers and zero). Evaluating the function at the arguments 0, 1, 2 and 3, we find

$$f(0) = 4 + 0 - 0^2 = 4$$
$$f(1) = 4 + 1 - 1^2 = 4$$
$$f(2) = 4 + 2 - 2^2 = 2$$
$$f(3) = 4 + 3 - 3^2 = -2$$

We plot the four ordered pairs $(0, 4)$, $(1, 4)$, $(2, 2)$ and $(3, -2)$, and connect them with a smooth curve (Figure 1.2.13). This is precisely the method that calculators and computers use to plot functions, except that they generally use 20 or more points to make a graph.

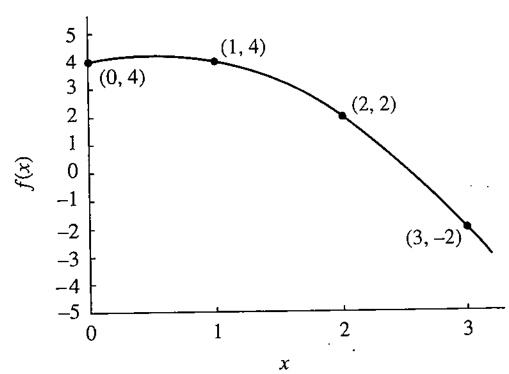


FIGURE 1.2.13
Plotting a function from its formula

One of the great advantages of functional notation is that functions can be evaluated at arguments that consist of parameters and variables (combinations of letters). To do so, replace the basic variable in the formula with the new argument, however complicated.

Example 1.2.8 Evaluating a Function at a Complicated Argument

To evaluate the function $f(x) = 4 + x - x^2$ (Example 1.2.7) at the more complicated argument $2z + 3$, replace all occurrences of x in the formula with the new argument $2z + 3$, obtaining

$$f(2z + 3) = 4 + (2z + 3) - (2z + 3)^2$$

To avoid confusion, place the new argument in parentheses wherever it appears. Although doing so is not always necessary, this expression can be multiplied out and simplified as follows:

$$\begin{aligned}
f(2z+3) &= 4+(2z+3)-(2z+3)^2 && \text{original expression}\\
&= 4+(2z+3)-\left(4z^2+12z+9\right) && \text{expand the square}\\
&= 4+2z+3-4z^2-12z-9 && \text{multiply negative sign through}\\
&= 4+3-9+2z-12z-4z^2 && \text{group like terms}\\
&= -2-10z-4z^2 && \text{combine like terms}
\end{aligned}$$

Example 1.2.9 A Function Describing Bacterial Population Growth

The population in Examples 1.2.1 and 1.2.2 obeys the formula

$$b(t)=\frac{t^2}{4.2}+1.0$$

The population size b is a function of the time t. The **argument** of the function b is t, the time after the beginning of the experiment. The **value** of the function is the population of bacteria. The formula summarizes the relation between these two measurements: The output is found by squaring the input, dividing by 4.2, and then adding 1.0.

The function b takes time after the beginning of the experiment as its input. Because negative time does not make sense in this case, the **domain** of this function consists of all positive numbers and zero. We write that

$$b \text{ is defined on the domain } t \geq 0$$

Because the function b returns population sizes as output, the **codomain** of b also consists of all positive numbers and zero. We write that

$$b \text{ has codomain } b \geq 0$$

The **range** is $b \geq 1$.

Example 1.2.10 A Function with Non-Numerical Domain

Animal	Number of Legs
Ant	6
Crab	10
Duck	2
Fish	0
Human	2
Mouse	4
Spider	8

Consider the adjacent table of data. These data describe a relation between two observations: the identity of the species and the number of legs. We can express this as the function L (to remind us of legs). According to the table,

$$L(\text{Ant})=6,\quad L(\text{Crab})=10$$

and so forth. The domain of this function is "types of animals," and the codomain is the non-negative integers $(0, 1, 2, 3, \ldots)$. We plot the input ("animal") along the horizontal axis and the output ("number of legs") on the vertical axis (Figure 1.2.14).

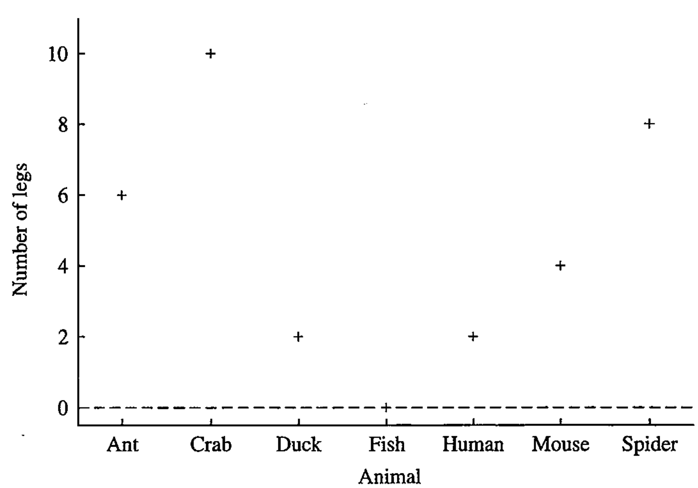


FIGURE 1.2.14
Numbers of legs on various organisms, plotted on a graph

It is important to realize that the graph of a function is *not* the function, just as the spot labeled 2 on the number line is not the number 2 and a photograph of a dog is not a dog. The graph is a depiction of the function.

Functions can be described in four ways: (1) numerically (by means of a table), (2) algebraically (as a formula), (3) pictorially (as a graph), and (4) verbally. Biologists or applied mathematicians need to learn to use all four methods and to translate fluently between them. In particular, we must know how to translate graphical information into words that communicate key observations to colleagues and the public.

Example 1.2.11 Describing Results in Graphs and Words

A more complicated pattern of change in population size is presented in the adjacent table.

Time	Population Size
0	0.86
2	1.69
4	2.98
6	4.49
8	5.69
10	6.17
12	5.95
14	5.29
16	4.41
18	3.50
20	2.67
22	1.96
24	1.41

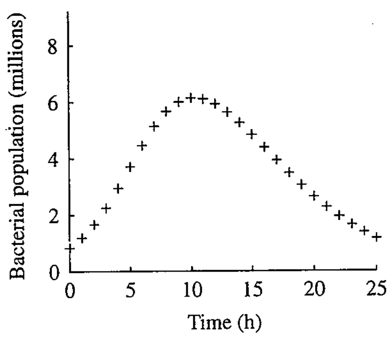


FIGURE 1.2.15
The population of bacteria in a culture

We can see (more easily from the graph, Figure 1.2.15, than from the table) that the bacterial population grew during the first ten hours and declined thereafter. The population reached a maximum at time 10. This graph and its description can be used to understand the results even without a mathematical formula.

Example 1.2.12 Sketching a Graph from a Verbal Description

Conversely, it can be useful to sketch a graph of a function from a verbal description. Suppose we are told that a population increases between time 0 and time 5, decreases nearly to 0 by time 12, increases to a higher maximum at time 20, and goes extinct at time 30. A graph (Figure 1.2.16) translates this information into pictorial form. Because we

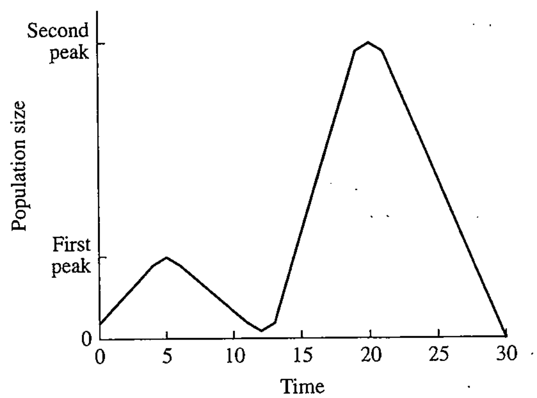


FIGURE 1.2.16
A bacterial population plotted from a verbal description

were not given exact values, the graph is not exact. It instead gives a **qualitative** picture of the results.

Not all relations are described by functions. A function must give a unique output for a given input. Relations between measurements can be more complicated.

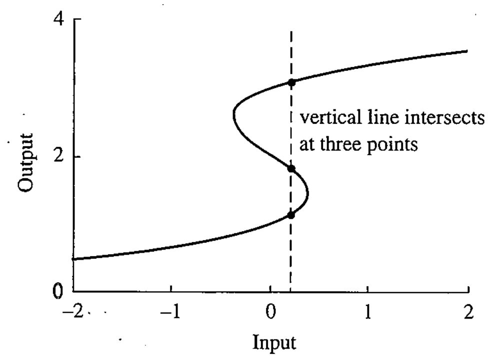


FIGURE 1.2.17
The vertical line test

The **vertical line test** provides a graphical method to recognize relations that cannot be described by functions.

The Vertical Line Test A relation is not a function if some vertical line crosses the graph two or more times.

In Figure 1.2.17, there are three outputs associated with the input 0.2: 1.12, 1.79, and 3.09.

There is nothing wrong with relations that cannot be described by functions. Experiments, even when performed under apparently identical conditions, rarely produce identical results. As we will see when we study statistics (Chapter 8), functions are a useful mathematical idealization of the expected or average result of an experiment.

Example 1.2.13 A Mathematical Formula Describing a Relation That Is Not a Function

The set of solutions for x and y satisfying the equation

$$x^2 + y^2 = 1$$

is the circle of radius 1 centered at the origin (Figure 1.2.18). Each value of x between $x = -1$ and $x = 1$ is associated with two different values of y. For example, the value $x = 0.6$ is associated with both $y = 0.8$ and $y = -0.8$.

Example 1.2.14 A Relation That Is Not a Function

Suppose several bacterial cultures with different initial population sizes are grown in controlled conditions for 1 hour, as in Example 1.2.3, with the results shown in the accompanying table and in Figure 1.2.19.

Colony	Initial Population, b_i	Final Population, b_f
1	0.5	0.9
2	0.5	1.0
3	1.0	2.2
4	1.0	1.9
5	1.5	3.0
6	1.5	2.8

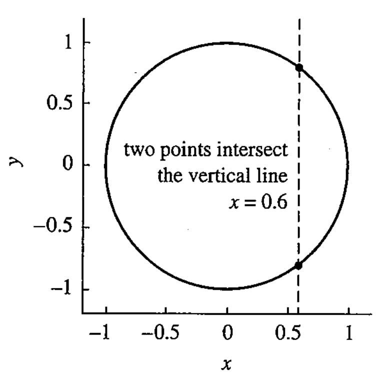


FIGURE 1.2.18
The circle describes a relation that is not a function

Each initial population was used twice, with similar but not identical results. We cannot treat final population size as a function of initial population size.

Combining Functions

Mathematics makes complicated problems simpler by building complicated structures from simple pieces. Understanding each of the simple pieces and the rules for

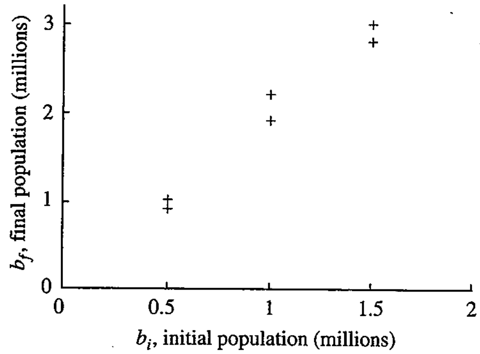


FIGURE 1.2.19
Bacterial growth experiment where results are not a function

combining them makes it possible to analyze and understand a huge array of complicated relations. The most important ways to combine functions are as **sums, products,** and **compositions.**

Adding Functions The height of the graph of the sum of two functions is the height of the first plus the height of the second. Geometrically, we can graph each of the pieces and add them together point by point.

Algebraically, the value of the function $f + g$ is computed as the sum of the values of the functions f and g.

Definition 1.3 The sum $f + g$ of the functions f and g is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

Multiplying Functions The value of the product $f \cdot g$ is computed as the product of the values of the functions f and g.

Definition 1.4 The product $f \cdot g$ of the functions f and g is the function defined by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

We use the dot $\cdot$ rather than the times sign $\times$ to indicate multiplication to avoid confusing the latter with the variable x.

Example 1.2.15 Adding and Multiplying Functions

Consider the functions $f(x)$ and $g(x)$ with formulas

$$f(x) = 4 + x - x^2$$
$$g(x) = 2x$$

graphed in Figures 1.2.20 and 1.2.21. The table on p. 14 computes the values of $f + g$ and $f \cdot g$ at several points.

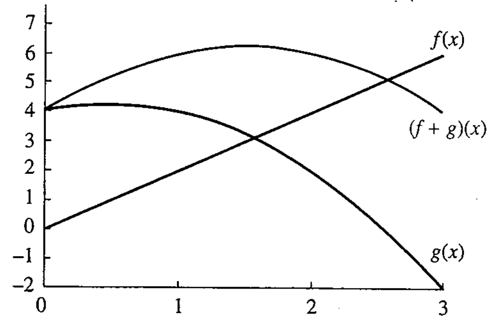


FIGURE 1.2.20
Adding functions

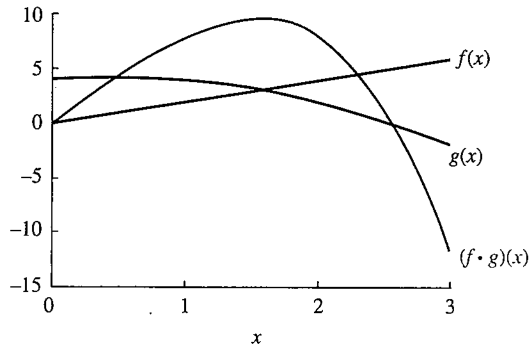


FIGURE 1.2.21
Multiplying functions

x	$f(x)$	$g(x)$	$(f+g)(x)$	$(f \cdot g)(x)$
0	4	0	4	0
0.5	4.25	1	5.25	4.25
1	4	2	6	8
1.5	3.25	3	6.25	9.75
2	2	4	6	8
2.5	0.25	5	5.25	1.25
3	−2	6	4	−12

Example 1.2.16 Adding Biological Functions

If two bacterial populations are separately counted, the total population is the sum of the two individual populations. Suppose a growing population is described by the function

$$b_1(t) = t^2 + 1$$

and a declining population is described by the function

$$b_2(t) = \frac{5}{1+2t}$$

The individual population sizes and their sum are computed in the following table and graphed in Figure 1.2.22.

t	$b_1(t)$	$b_2(t)$	$(b_1+b_2)(t)$
0.00	1.00	5.00	6.00
0.50	1.25	2.50	3.75
1.00	2.00	1.67	3.67
1.50	3.25	1.25	4.50
2.00	5.00	1.00	6.00
2.50	7.25	0.83	8.08
3.00	10.00	0.71	10.71

Example 1.2.17 Multiplying Biological Functions

Many quantities in science are built as products of simpler quantities. For example, the mass of a population is the product of the mass of each individual and the number of individuals. Consider a population growing according to

$$b(t) = \frac{t^2}{4.2} + 1.0$$

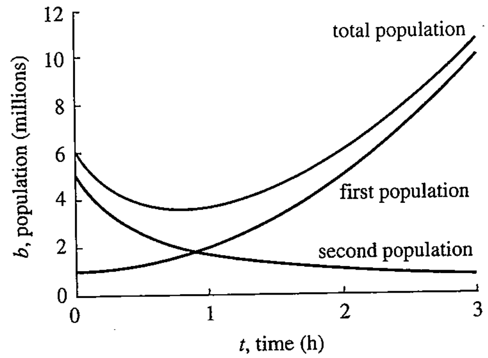


FIGURE 1.2.22
Adding biological functions

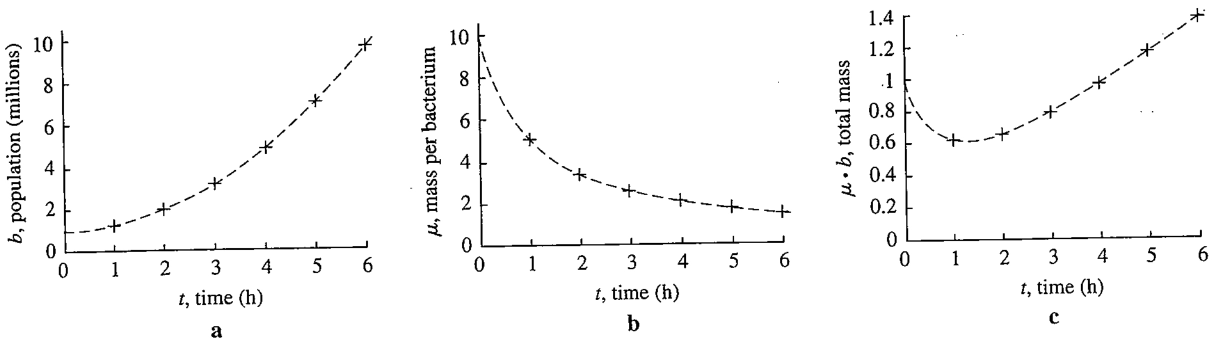


FIGURE 1.2.23
Multiplying biological functions

t	b	μ	$\mu \cdot b$
0.0	1.00	1.00	1.00
1.0	1.24	0.50	0.62
2.0	1.95	0.33	0.65
3.0	3.14	0.25	0.79
4.0	4.81	0.20	0.96
5.0	6.95	0.17	1.16
6.0	9.57	0.14	1.37

(Example 1.2.9). Suppose that as the population gets larger, the individuals become smaller. Let $\mu(t)$ (the Greek letter mu)[1] represent the mass of an individual at time t, and suppose that

$$\mu(t) = \frac{1}{1+t}$$

We can find the total mass by multiplying the mass per individual by the number of individuals, as in the table. (See also Figure 1.2.23.)

The total mass of this population initially declines and then increases after about 2 hours.

Composition of Functions The most important way to combine functions is through **composition,** where the output of one function acts as the input of another.

Definition 1.5 The composition $f \circ g$ of functions f and g is the function defined by

$$(f \circ g)(x) = f(g(x)) \tag{1.2.1}$$

We say "f composed with g evaluated at x" or "f of g of x." The function f is called the **outer function,** and g is called the **inner function.** See Figure 1.2.24.

Example 1.2.18 Computing the Value of a Functional Composition

Consider the functions $f(x)$ and $g(x)$ from Example 1.2.15,

$$f(x) = 4 + x - x^2$$
$$g(x) = 2x$$

[1] Applied mathematicians often use Greek letters to represent variables and parameters. The Greek alphabet, along with pronunciations of the letters, is given on the inside back cover.

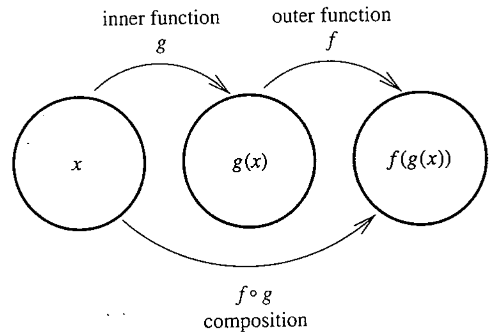


FIGURE 1.2.24
Composition of functions

To find the value of the composition $f \circ g$ at $x = 2$, we compute

$(f \circ g)(2) = f(g(2))$	definition of composition
$= f(2 \cdot 2)$	substitute 2 for x in the formula for $g(x)$
$= f(4)$	compute $2 \cdot 2 = 4$
$= 4 + 4 - 4^2$	substitute 4 for x in the formula for $f(x)$
$= -8$	compute the numerical answer

Similarly, to find the value of the composition $g \circ f$ at $x = 2$, we compute

$(g \circ f)(2) = g(f(2))$	definition of composition
$= g(4 + 2 - 2^2)$	substitute 2 for x in the formula for $f(x)$
$= g(2)$	compute $4 + 2 - 2^2 = 2$
$= 2 \cdot 2$	substitute 2 for x in the formula for $g(x)$
$= 4$	compute the numerical answer

Example 1.2.19 Computing the Formula of a Functional Composition

Consider again the functions $f(x)$ and $g(x)$ from Example 1.2.18,

$$f(x) = 4 + x - x^2$$
$$g(x) = 2x$$

with domains consisting of all numbers. To find the composition $f \circ g$, plug the definition of the **inner function** g into the formula for the **outer function** f, or

$(f \circ g)(x) = f(g(x))$	the definition
$= f(2x)$	write out the formula for the inner function $g(x)$
$= 4 + (2x) - (2x)^2$	plug the formula for $g(x)$ into the outer function f
$= 4 + 2x - 4x^2$	expand the square

This is the same procedure we used to compute the value of the function $f(x)$ at a complicated argument in Example 1.2.8. In Example 1.2.18 we computed that $(f \circ g)(2) = -8$. If we evaluate by substituting into the formula $(f \circ g)(x) = 4 + 2x - 4x^2$, we find

$$(f \circ g)(2) = 4 + 2 \cdot 2 - 4 \cdot 2^2 = -8$$

matching our earlier result.

We find the composition $g \circ f$ by following the same steps, or

$(g \circ f)(x) = g(f(x))$	the definition
$= g(4 + x - x^2)$	write out the formula for the inner function $f(x)$
$= 2(4 + x - x^2)$	plug the formula for $f(x)$ into the outer function g
$= 8 + 2x - 2x^2$	multiply through

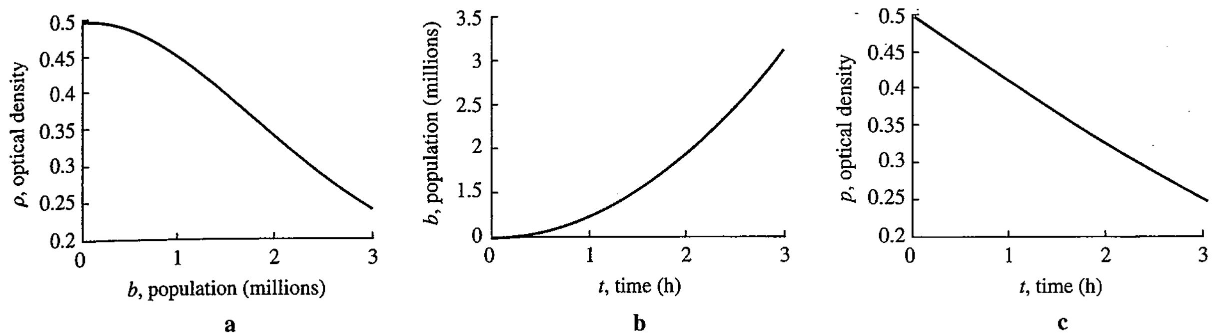


FIGURE 1.2.25
Composing biological functions

The key step is substituting the output of the inner function into the outer function. In Example 1.2.18 we computed that $(g \circ f)(2) = 4$. If we evaluate by substituting into the formula $(g \circ f)(x) = 8 + 2x - 2x^2$, we find

$$(g \circ f)(2) = 8 + 2 \cdot 2 - 2 \cdot 2^2 = 4$$

again matching our earlier result.

Example 1.2.19 illustrates an important point about the composition of functions: the answer is generally different when the functions are composed in a different order. If $f \circ g = g \circ f$, we say that the two functions **commute.** When the two compositions do not match, we say that the two functions do not commute. Without a good reason, never assume that two functions commute. If you think of functions as operations, this should make sense. Sterilizing the scalpel and then making an incision produces a quite different result from making an incision and then sterilizing the scalpel.

Example 1.2.20 Composition of Functions in Biology

Numbers of bacteria are usually measured indirectly, by measuring the optical density of the medium. Water allows through less light as the population becomes larger. Suppose that the optical density ρ is a function of the bacterial population size b with formula

$$\rho(b) = \frac{1}{1+b}$$

t	$b(t)$	$\rho(b(t))$
0.00	1.00	0.500
0.50	1.06	0.486
1.00	1.24	0.447
1.50	1.54	0.394
2.00	1.95	0.339
2.50	2.49	0.287
3.00	3.14	0.241

illustrated in Figure 1.2.25a. Then the optical density as a function of time is the composition of the function $\rho(b)$ with the function $b(t)$. Suppose that $b(t) = \frac{t^2}{4.2} + 1.0$ as in Example 1.2.9 (Figure 1.2.25b). Then

$$\rho(b(t)) = \rho\left(\frac{t^2}{4.2} + 1.0\right) = \frac{1}{1 + \frac{t^2}{4.2} + 1.0}$$

with values given in the table and graphed in Figure 1.2.25c.

The composition $b \circ \rho$ is not merely different from the composition $\rho \circ b$, it does not even make sense. The function b accepts as input only the time t, not the optical density returned as output by the function ρ. We will study this issue more carefully in Section 1.3.

Finding Inverse Functions

A function describes the relation between two measurements and gives us a way to compute the output from a given input. Sometimes we wish to reverse the process and figure out which input produced a given output. The **inverse function,** when it exists, provides a way to do this.

Example 1.2.21 A Simple Inverse Operation

What number, when doubled, gives 8? It is not difficult to guess that the answer is 4. However, we can formalize this process using functional notation. Let $f(x) = 2x$ be the function that doubles. Our problem is then solving

$$f(x) = 8$$

Using the formula for $f(x)$, we find

$$\begin{aligned} 2x &= 8 && \text{the equation to be solved} \\ x &= 4 && \text{divide both sides by 2} \end{aligned}$$

Example 1.2.22 A Simple Inverse Function

Example 1.2.21 undoes the act of multiplying by 2. What function does this in general? If we set $y = f(x)$, we would like to know what value of x produces a given y in general, without picking a particular value such as $y = 8$. We follow the same steps,

$$\begin{aligned} 2x &= y && \text{the equation to be solved} \\ x &= \frac{y}{2} && \text{divide both sides by 2} \end{aligned}$$

The function f^{-1}, which is read "f inverse" and defined by

$$f^{-1}(y) = \frac{y}{2}$$

is the **inverse** of f; the function that undoes what f did in the first place. Whereas f takes a number as input and returns double that number as output, f^{-1} takes the doubled number as input and returns the initial number as output.

We can use this inverse like any other function, finding that

$$f^{-1}(8) = \frac{8}{2} = 4$$

as we found in Example 1.2.21.

The definition of an inverse function in general states precisely that the inverse undoes the action of the original function.

Definition 1.6 The function f^{-1} is the inverse of f if

$$f\left(f^{-1}(x)\right) = x$$

and

$$f^{-1}(f(x)) = x$$

Each of f and f^{-1} undoes the action of the other (Figure 1.2.26).

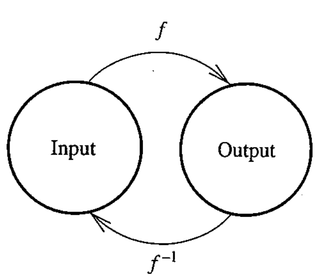


FIGURE 1.2.26
The action of a function and its inverse

The steps for computing the inverse of a function can be summarize in an **algorithm,** which can be thought of as a recipe. This book contains many algorithms for solving particular problems. As with a recipe, following an algorithm without thinking about the steps can lead to disaster. Unlike most algorithms in this book, this one is not guaranteed to work.

Algorithm 1.1 Finding the Inverse of a Function

1. Write the equation $y = f(x)$.
2. Solve for x in terms of y.
3. The inverse function is the operation done to y.

It may look odd to have a function defined in terms of y. Do *not* change the letters around to make it look normal. In applied mathematics, different letters stand for different things and resent having their names switched as much as we do.

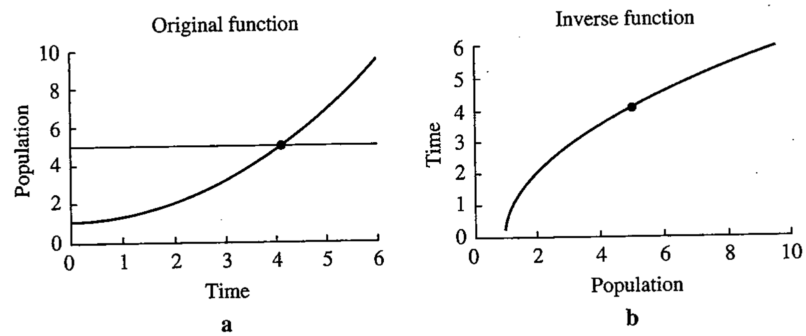


FIGURE 1.2.27
Going backwards with the inverse function

This algorithm may fail in two different ways: a function might not have an inverse, or the inverse might be impossible to compute. There is a useful way to recognize a function that fails to have an inverse. An operation can be undone only if you can deduce the input from the output. If any particular output is associated with more than one input, there is no way to tell where you started solely on the basis of where you ended up.

Example 1.2.23 Finding a More Complicated Inverse

Consider the population that changes in accordance with the equation

$$b(t) = \frac{t^2}{4.2} + 1.0$$

(Example 1.2.9 and Figure 1.2.27a). If we wish to find the time t from the population b, we must solve for t.

$\frac{t^2}{4.2} + 1.0 = b$	the equation to be solved for t
$\frac{t^2}{4.2} = b - 1.0$	subtract 1.0 from both sides
$t^2 = 4.2(b - 1.0)$	multiply both sides by 4.2
$t = \sqrt{4.2(b - 1.0)}$	take the positive square root of both sides because $t \geq 0$

This function is graphed in Figure 1.2.27b. The last step requires that $b \geq 1.0$ because we cannot take the square root of a negative number. For example, the time associated with a population of 5.0 is

$$t = \sqrt{4.2(5.0 - 1.0)} \approx 4.1$$

Example 1.2.24 A Relation That Cannot Be Inverted

Consider the data in the following table.

Initial Mass (g)	Final Mass (g)	Initial Mass (g)	Final Mass (g)
1.0	7.0	9.0	20.0
2.0	12.0	10.0	18.0
3.0	16.0	11.0	15.0
4.0	19.0	12.0	12.0
5.0	22.0	13.0	9.0
6.0	23.0	14.0	6.0
7.0	23.0	15.0	3.0
8.0	22.0	16.0	1.0

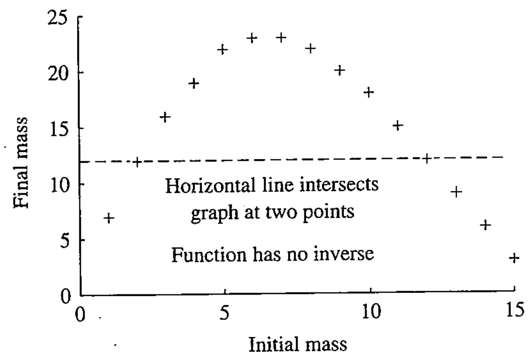


FIGURE 1.2.28
A relation with no inverse

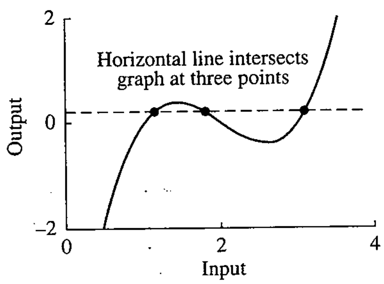


FIGURE 1.2.29
The horizontal line test

Suppose you were told that the mass at the end of the experiment was 12.0 grams. Initial masses of 2.0 and 12.0 grams both produce a final mass of 12.0 grams. You cannot tell whether the input was 2.0 or 12.0. This function has no inverse (Figure 1.2.28).

This reasoning leads to a useful graphical test for whether a function has an inverse.

The Horizontal Line Test A function has no inverse if it takes on the same value twice. This can be established by graphing the function and checking whether the graph intersects any horizontal line two or more times (Figure 1.2.29).

One can think of functions without inverses as losing information over the course of the experiment: things that started out different ended up the same.

Example 1.2.25 A Function That Has an Inverse on Part of Its Domain

Consider the function $g(x) = x^2$ defined for $x \geq 0$ (Figure 1.2.30). We find the inverse $f^{-1}(y)$ by solving $y = x^2$ for x.

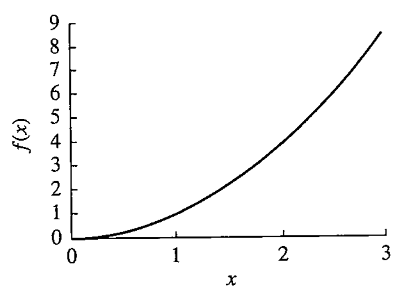


FIGURE 1.2.30
The inverse of x^2 is defined when $x \geq 0$

1. Set $y = x^2$.
2. Then $x = \sqrt{y}$ because $x \geq 0$.
3. $f^{-1}(y) = \sqrt{y}$.

Example 1.2.26 A Function Without an Inverse

Consider the function $f(x) = 4 + x - x^2$ (used in Example 1.2.7). We found that the inputs $x = 0$ and $x = 1$ both produce the same output of $f(x) = 4$ (Figure 1.2.31). If the

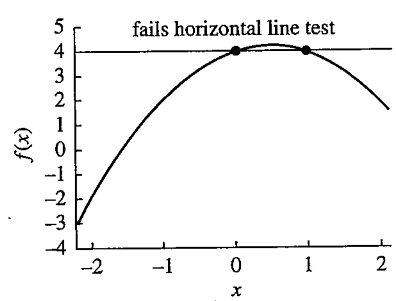


FIGURE 1.2.31
A function with no inverse

output is 4, it is impossible to tell which was the input. A graph shows that this function fails the horizontal line test at almost all values in its range. ▲

In addition, Algorithm 1.1 might fail because the algebra is impossible. Step 2 requires solving an equation. Many equations cannot be solved algebraically.

Example 1.2.27 A Function with an Inverse That Is Impossible to Compute Algebraically

Consider the function

$$f(x) = x^5 + x + 1$$

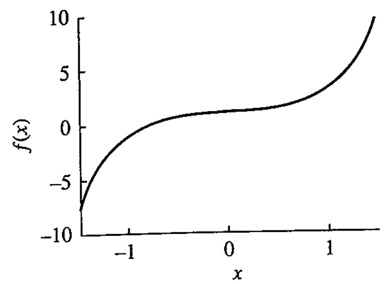


FIGURE 1.2.32
A function with an inverse that is impossible to compute

The graph satisfies the horizontal line test (see Figure 1.2.32). We try to find the inverse $f^{-1}(y)$ as follows:

1. Set $y = x^5 + x + 1$.
2. Try to solve for x. Even with the cleverest algebraic tricks, this is impossible (a remarkable theorem, proved by the French mathematician Evariste Galois when he was just 20 years old, assures us that there is no formula for the solution of a general polynomial with degree greater than 4).
3. Give up. ▲

In mathematical modeling, however, it is often more important to know that something exists (such as the inverse in this case) than to be able to write down a formula. We will later learn a method to compute this inverse numerically, with a computer (Section 3.8).

Summary Quantitative science is built upon measurements, and mathematics provides the methods for describing and thinking about measurements and relations between them. **Variables** describe measurements that change during the course of an experiment, and **parameters** describe measurements that remain constant during an experiment but might change between different experiments. **Functions** describe relations between different measurements when a single ouput is associated with each input; they can be recognized graphically with the **vertical line test.** New functions are built by combining functions through **addition, multiplication,** and **composition.** In functional composition, the output of the **inner function** is used as the input of the **outer function.** Many functions do not **commute,** meaning that composing the functions in a different order gives a different result. Finally, we can use the **horizontal line test** to check whether a functions has an **inverse.** If it does, the inverse can be used to compute the input from the output.

1.2 Exercises

Mathematical Techniques

1–2 ▪ Give mathematical names to the measurements in the following situations, and identify the variables and parameters.

1. A scientist measures the density of wombats at three altitudes: 500 m, 750 m, and 1000 m. He repeats the experiment in 3 different years, with rainfall of 30 cm in the first year, 50 cm in the second, and 60 cm in the third.
2. A scientist measures the density of bandicoots at three altitudes: 500 m, 750 m, and 1000 m. She repeats the experiment in three different years that have different densities of wombats, which compete with bandicoots. The density is 10 wombats per square kilometer in the first year, 20 wombats per square kilometer in the second, and 15 wombats per square kilometer in the third.

3–6 ▪ Compute the values of the following functions at the points indicated and sketch a graph.

3. $f(x) = x + 5$ at $x = 0$, $x = 1$, and $x = 4$
4. $g(y) = 5y$ at $y = 0$, $y = 1$, and $y = 4$
5. $h(z) = \frac{1}{5z}$ at $z = 1$, $z = 2$, and $z = 4$
6. $F(r) = r^2 + 5$ at $r = 0$, $r = 1$, and $r = 4$

7–10 ▪ Graph the given points and say which point does not seem to fall on the graph of a simple function that describes the other four.

7. (0, −1), (1, 1), (2, 2), (3, 5), (4, 7)
8. (0, 8), (1, 10), (2, 8), (3, 6), (4, 4)
9. (0, 2), (1, 3), (2, 6), (3, 11), (4, 12)
10. (0, 30), (1, 25), (2, 15), (3, 12), (4, 10)

11–14 ▪ Evaluate the following functions at the given algebraic arguments.

11. $f(x) = x + 5$ at $x = a$, $x = a + 1$, and $x = 4a$
12. $g(y) = 5y$ at $y = x^2$, $y = 2x + 1$, and $y = 2 - x$
13. $h(z) = \frac{1}{5z}$ at $z = \frac{c}{5}$, $z = \frac{5}{c}$, and $z = c + 1$
14. $F(r) = r^2 + 5$ at $r = x + 1$, $r = 3x$, and $r = \frac{1}{x}$

15–16 ▪ Sketch graphs of the following relations. Is there a more convenient order for the arguments?

15. A function whose argument is the name of a state and whose value is the highest altitude in that state.

State	Highest Altitude (ft)
California	14,491
Idaho	12,662
Nevada	13,143
Oregon	11,239
Utah	13,528
Washington	14,410

16. A function whose argument is the name of a bird and whose value is the average length of that bird.

Bird	Length (cm)
Cooper's hawk	50
Goshawk	66
Sharp-shinned hawk	35

17–20 ▪ For each of the following pairs of functions, graph each component piece. Compute the value of the sum at $x = -2$, $x = -1$, $x = 0$, $x = 1$, and $x = 2$ and plot the result.

17. $f(x) = 2x + 3$ and $g(x) = 3x - 5$
18. $f(x) = 2x + 3$ and $h(x) = -3x - 12$
19. $F(x) = x^2 + 1$ and $G(x) = x + 1$
20. $F(x) = x^2 + 1$ and $H(x) = -x + 1$

21–24 ▪ For each of the following pairs of functions, graph each component piece. Compute the value of the product at $x = -2$, $x = -1$, $x = 0$, $x = 1$, and $x = 2$ and graph the result.

21. $f(x) = 2x + 3$ and $g(x) = 3x - 5$
22. $f(x) = 2x + 3$ and $h(x) = -3x - 12$
23. $F(x) = x^2 + 1$ and $G(x) = x + 1$
24. $F(x) = x^2 + 1$ and $H(x) = -x + 1$

25–28 ▪ Find the inverse of each of the following functions when an inverse exists. In each case, compute the output at an input of 1.0, and show that the inverse undoes the action of the function.

25. $f(x) = 2x + 3$
26. $g(x) = 3x - 5$
27. $F(y) = y^2 + 1$
28. $F(y) = y^2 + 1$ for $y \geq 0$

29–32 ▪ Graph each of the following functions and its inverse if it exists. Mark the given point on the graph of each function.

29. $f(x) = 2x + 3$. Mark the point $(1, f(1))$ on the graphs of f and the corresponding point on f^{-1} (based on Exercise 25).
30. $g(x) = 3x - 5$. Mark the point $(1, g(1))$ on the graphs of g and the corresponding point on g^{-1} (based on Exercise 26).
31. $F(y) = y^2 + 1$. Mark the point $(1, F(1))$ on the graphs of F and the corresponding point on F^{-1} (based on Exercise 27).
32. $F(y) = y^2 + 1$ for $y \geq 0$. Mark the point $(1, F(1))$ on the graphs of F and the corresponding point on F^{-1} (based on Exercise 28).

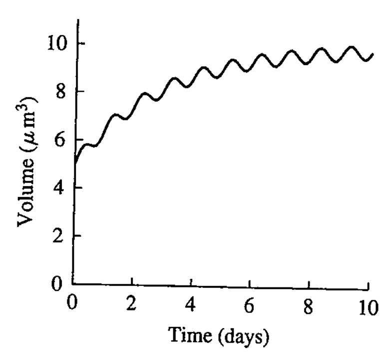
33–36 ▪ Find the compositions of the given functions. Which pairs of functions commute?

33. $f(x) = 2x + 3$ and $g(x) = 3x - 5$
34. $f(x) = 2x + 3$ and $h(x) = -3x - 12$
35. $F(x) = x^2 + 1$ and $G(x) = x + 1$
36. $F(x) = x^2 + 1$ and $H(x) = -x + 1$

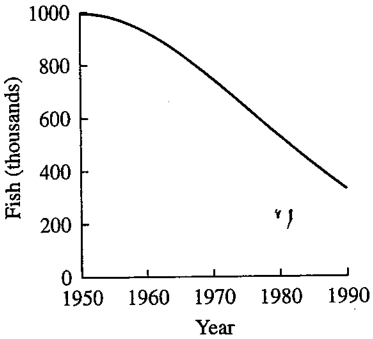
Applications

37–40 ▪ Describe what is happening in the graphs shown.

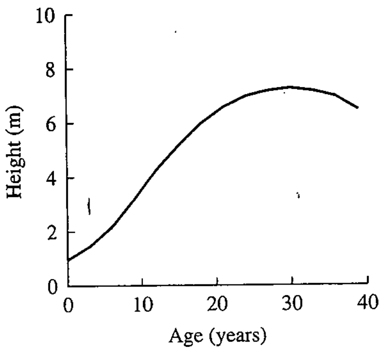
37. A plot of cell volume against time in days.



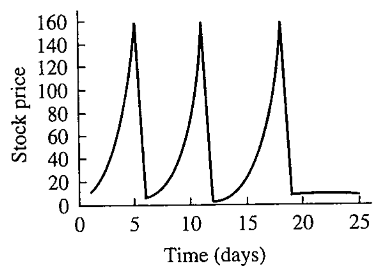
38. A plot of a Pacific salmon population against time in years.



39. A plot of the average height of a population of trees plotted against age in years.



40. A plot of an Internet stock price against time.



41–44 ▪ Draw graphs based on the following descriptions.

41. A population of birds begins at a large value, decreases to a tiny value, and then increases again to an intermediate value.

42. The amount of DNA in an experiment increases rapidly from a very small value and then levels out at a large value before declining rapidly to 0.

43. Body temperature oscillates between high values during the day and low values at night.

44. Soil is wet at dawn, quickly dries out and stays dry during the day, and then becomes gradually wetter again during the night.

45–48 ▪ Evaluate the following functions over the suggested range, sketch a graph of the function, and answer the biological question.

45. The number of bees b found on a plant is given by $b = 2f + 1$ where f is the number of flowers, ranging from 0 to about 20. Explain what might be happening when $f = 0$.

46. The number of cancerous cells c as a function of radiation dose r (measured in rads) is

$$c = r - 4$$

for r greater than or equal to 5, and is zero for r less than 5. Suppose r ranges from 0 to 10. What is happening at $r = 5$ rads?

47. Insect development time A (in days) obeys $A = 40 - \frac{T}{2}$ where T represents temperature in °C for $10 \le T \le 40$. Which temperature leads to the most rapid development?

48. Tree height h (in meters) follows the formula

$$h = \frac{100a}{100 + a}$$

where a represents the age of the tree in years. The formula is valid for any positive value of a, which ranges from 0 to 1000. How tall would this tree get if it lived forever?

49–52 ▪ Consider the following data describing the growth of an tadpole.

Age, a (days)	Length, L (cm)	Tail Length, T (cm)	Mass M (g)
0.5	1.5	1.0	1.5
1.0	3.0	0.9	3.0
1.5	4.5	0.8	6.0
2.0	6.0	0.7	12.0
2.5	7.5	0.6	24.0
3.0	9.0	0.5	48.0

49. Graph length as a function of age.

50. Graph tail length as a function of age.

51. Graph tail length as a function of length.

52. Graph mass as a function of length, and then graph length as a function of mass. How do the two graphs compare?

53–56 ▪ The following series of functional compositions describe connections between several measurements.

53. The number of mosquitos (M) that end up in a room is a function of how much the window is open (W, in square centimeters) according to $M(W) = 5W + 2$. The number of bites (B) depends on the number of mosquitos according to $B(M) = 0.5M$. Find the number of bites as a function of how much the window is open. How many bites would you get if the window were 10 cm^2 open?

54. The temperature of a room (T, in degrees Celsius) is a function of how much the window is open (W, in square centimeters) according to $T(W) = 40 - 0.2W$. How long you sleep (S, measured in hours) is a function of the temperature according to $S(T) = 14 - \frac{T}{5}$. Find how long you sleep as a function of how much the window is open. How long would you sleep if the window were 10 cm^2 open?

55. The number of viruses (V, measured in trillions) that infect a person is a function of the degree of immunosuppression (I, the fraction of the immune system that is turned off by medication) according to $V(I) = 5I^2$. The fever (F, measured in °C) associated with an infection is a function of the number of viruses according to $F(V) = 37 + 0.4V$. Find fever as a function of immunosuppression. How high will the fever be if immunosuppression is complete ($I = 1$)?

56. The length of an insect (L, in millimeters) is a function of the temperature during development (T, measured in °C) according to $L(T) = 10 + \frac{T}{10}$. The volume of the insect (V, in cubic millimeters) is a function of the length according to $V(L) = 2L^3$. The mass (M in milligrams) depends on volume according to $M(V) = 1.3V$. Find mass as a function of temperature. How much would an insect weigh that developed at 25°C? Would you be frightened to meet this insect?

57–58 ▪ Each of the following measurements is the sum of two components. Find the formula for the sum. Sketch a graph of each component and the total as functions of time for $0 \le t \le 3$. Describe each component and the sum in words.

57. A population of bacteria consists of two types, a and b. The first follows $a(t) = 1 + t^2$, and the second follows $b(t) = 1 - 2t + t^2$ where populations are measured in millions and time is measured in hours. The total population is $P(t) = a(t) + b(t)$.

58. The above-ground volume (stem and leaves) of a plant is $V_a(t) = 3.0t + 20.0 + \frac{t^2}{2}$ and the below-ground volume (roots) is $V_b(t) = -1.0t + 40.0$ where t is measured in days after seed germination and volumes are measured in cm^3. The domain is $0 \le t \le 40$. The total volume is $V(t) = V_a(t) + V_b(t)$.

59–62 ▪ Consider the following data describing a plant.

Age, a (days)	Mass, M (g)	Volume, V (cm^3)	Glucose production, G (mg)
0.5	1.5	5.1	0.0
1.0	3.0	6.2	3.4
1.5	4.3	7.2	6.8
2.0	5.1	8.1	8.2
2.5	5.6	8.9	9.4
3.0	5.6	9.6	8.2

59. Graph M as a function of a. Does this function have an inverse? Could we use mass to figure out the age of the plant?

60. Graph V as a function of a. Does this function have an inverse? Could we use volume to figure out the age of the plant?

61. Graph G as a function of a. Does this function have an inverse? Could we use glucose production to figure out the age of the plant?

62. Graph G as a function of M. Does this function have an inverse? What is strange about it? Could we use glucose production to figure out the mass of the plant?

63–66 ▪ The total mass of a population (in kg) as a function of the number of years, t, is the product of the number of individuals, $P(t)$, and the mass per person, $W(t)$ (in kg). In each of the following exercises, find the formula for the total mass, sketch graphs of $P(t)$, $W(t)$, and the total mass as functions of time for $0 \le t \le 100$, and describe the results as words.

63. The population of people P is $P(t) = 2.0 \times 10^6 + 2.0 \times 10^4 t$, and the mass per person $W(t)$ (in kg) is $W(t) = 80 - 0.5t$.

64. The population P is $P(t) = 2.0 \times 10^6 - 2.0 \times 10^4 t$, and the mass per person $W(t)$ is $W(t) = 80 + 0.5t$.

65. The population P is $P(t) = 2.0 \times 10^6 + 1000t^2$, and the mass per person $W(t)$ is $W(t) = 80 - 0.5t$.

66. The population P is $P(t) = 2.0 \times 10^6 + 2.0 \times 10^4 t$, and the mass per person $W(t)$ is $W(t) = 80 - 0.005t^2$.

Computer Exercises

67–70 ▪ Have your graphics calculator or computer plot the following functions. How would you describe them in words?

67. a. $f(x) = x^2 e^{-x}$ for $0 \le x \le 20$

b. $g(x) = 1.5 + e^{-0.1x} \sin(x)$ for $0 \le x \le 20$

c. $h(x) = \sin(5x) - \cos(7x)$ for $0 \le x \le 20$ for x measured in radians

d. $f(x) + h(x)$ for $0 \le x \le 20$ (using the functions in parts a and c)

e. $g(x) \cdot h(x)$ for $0 \le x \le 20$ (using the functions in parts b and c)

f. $h(x) \cdot h(x)$ for $0 \le x \le 20$ (using the function in part c)

68. Have your computer plot the function

$$h(x) = e^{-x^2} - e^{-1000(x-0.13)^2} - 0.2$$

for values of x between -10 and 10.

a. How would you describe the result in words?

b. Blow up the graph by changing the range to find all points where the value of the function is 0. For example, one such value is between 1 and 2. Plot the function again for x between 1 and 2 to zoom in

c. If you found only two points where $h(x) = 0$, blow up the region between 0 and 1 to try to find two more

69. Use your computer to find and plot the following functional compositions.

a. $(f \circ g)(x)$ and $(g \circ f)(x)$ if $f(x) = \sin(x)$ and $g(x) = x^2$

b. $(f \circ g)(x)$ and $(g \circ f)(x)$ if $f(x) = e^x$ and $g(x) = x^2$

c. $(f \circ g)(x)$ and $(g \circ f)(x)$ if $f(x) = e^x$ and $g(x) = \sin(x)$

Answers to Selected Odd Exercises

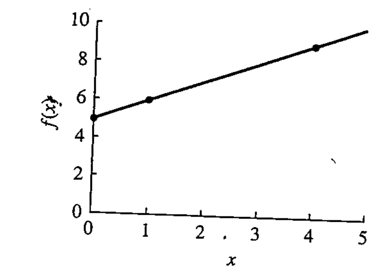
Answers to all odd exercises can be found at
http://www.brookscole.com/cgi-wadsworth/course_products_wp.pl?fid=M20b&product_isbn_issn=0534404863&discipline_number=1

Chapter 1

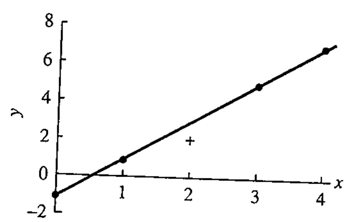
Section 1.2, page 21

1. The variables are the altitude and the wombat density, which we can call a and w, respectively. The parameter is the rainfall, which we can call R.

3. $f(0)=5$, $f(1)=6$, $f(4)=9$.

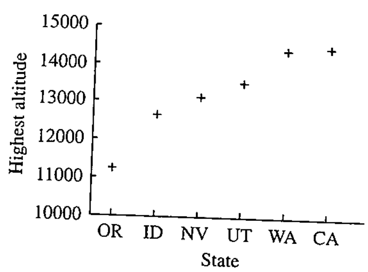


7. (2, 2).



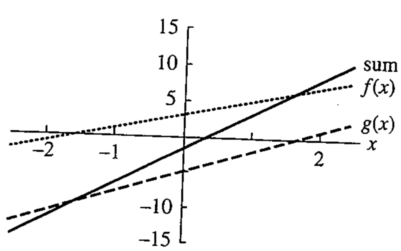
11. $f(a)=a+5$, $f(a+1)=a+6$, $f(4a)=4a+5$.

15. I put them in increasing order to look nice.



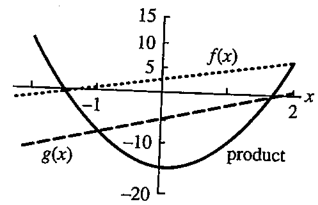
17.

x	$f(x)$	$g(x)$	$(f+g)(x)$
−2	−1	−11	−12
−1	1	−8	−7
0	3	−5	−2
1	5	−2	3
2	7	1	8



21.

x	$f(x)$	$g(x)$	$(f\cdot g)(x)$
−2	−1	−11	11
−1	1	−8	−8
0	3	−5	−15
1	5	−2	−10
2	7	1	7



25. If we write $y=2x+3$, we can solve for x with the steps

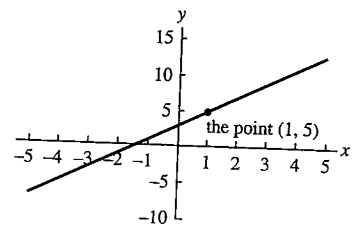
$$y-3=2x \qquad \text{subtract 3 from both sides}$$
$$\frac{y-3}{2}=x \qquad \text{divide both sides by 2.}$$

Therefore $f^{-1}(y)=\frac{y-3}{2}$. Also, $f(1)=5$, and $f^{-1}(5)=\frac{5-3}{2}=1$.

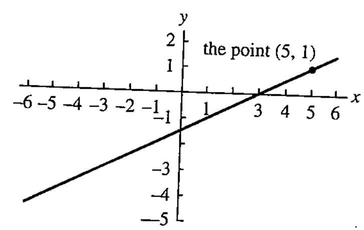
27. The function $F(y)$ fails the horizontal line test because, for example, $F(-1)=F(1)=2$. Therefore it has no inverse.

29.

The function

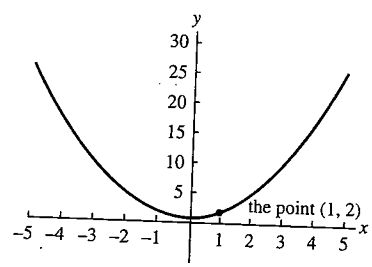


The inverse

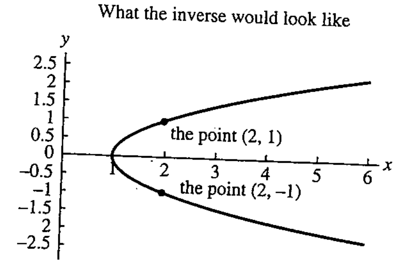


31. This function doesn't have an inverse because it fails the horizontal line test. From the graph, we couldn't tell whether $F^{-1}(2)$ is 1 or −1.

The function



What the inverse would look like

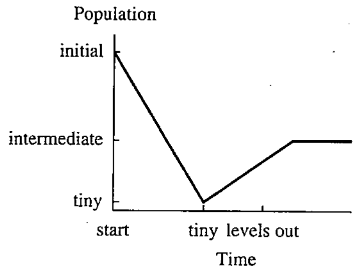


33. $(f\circ g)(x)=f(3x-5)=2\cdot(3x-5)+3=6x-7$ and $(g\circ f)(x)=g(2x+3)=3\cdot(2x+3)-5=6x+4$. These don't match, so the functions do not commute.

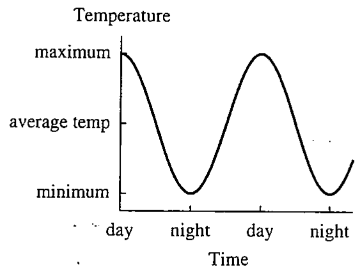
37. The cell volume is generally increasing but decreases during part of its cycle. The cell might get smaller when it gets ready to divide or during the night.

39. The height increases up until about age 30 and then decreases.

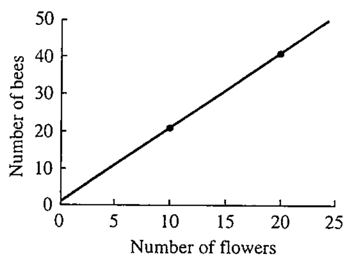
41.



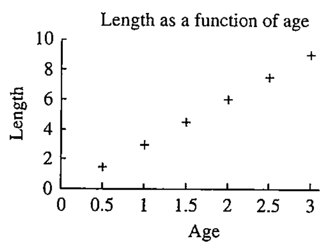
43.



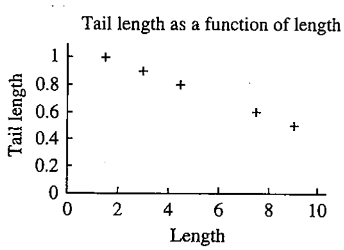
45. When $f = 0, b = 1$; when $f = 10, b = 21$; when $f = 20, b = 41$. Perhaps one bee will check out the plant even if there are no flowers.



49.



51.

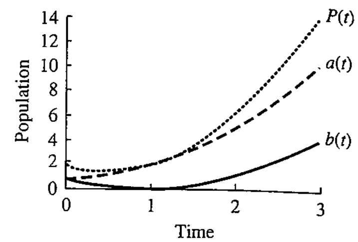


53. $B(M) = B(5W + 2) = 2.5W + 1$. Plugging in $W = 10$ gives 26 bites.

55. $F(V(I)) = F(5I^2) = 37 + 2I^2$. The fever is 39°C if $I = 1$.

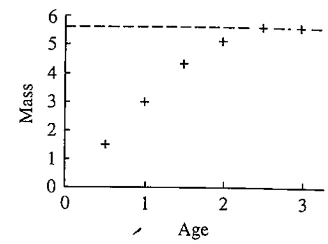
57. The formula is $P(t) = (1 + t^2) + (1 - 2t + t^2) = 2 - 2t + 2t^2$.

t	$a(t)$	$b(t)$	$P(t)$
0.00	1.00	1.00	2.00
0.50	1.25	0.25	1.50
1.00	2.00	0.00	2.00
1.50	3.25	0.25	3.50
2.00	5.00	1.00	6.00
2.50	7.25	2.25	9.50
3.00	10.00	4.00	14.00



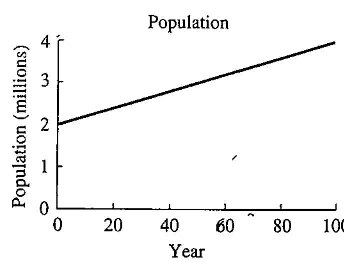
Here a is increasing, b decreases to 0 at time 1.0 and then increases, and the sum P decreases slightly and then increases.

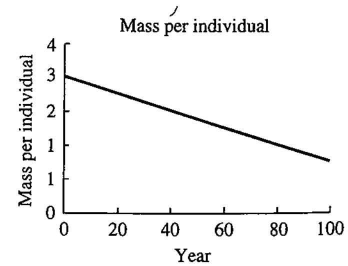
59. Because the mass is the same at ages 2.5 days and 3.0 days, the function relating a and M has no inverse. Knowing the mass does not give us enough information to estimate the age.

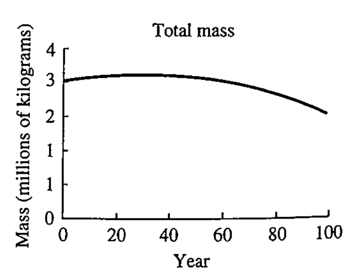


63. Denote the total mass by $T(t)$. Then $T(t) = P(t)W(T) = (2.0 \times 10^6 + 2.0 \times 10^4 t)(80 - 0.5t)$. Measuring population in millions gives

t	$P(t)$	$W(t)$	$T(t)$
0.0	2.0	80.0	160.0
20.0	2.4	70.0	168.0
40.0	2.8	60.0	168.0
60.0	3.2	50.0	160.0
80.0	3.6	40.0	144.0
100.0	4.0	30.0	120.0







The population increases, the mass per individual decreases, and the total mass increases and then decreases.