Review# 2 - MATH 180-009/013 - FALL 2008
December 05, 2008

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The final exam will evaluate material in Chapters 1-4. Here are the main points that we have discussed this semester. These correspond to Sections 1.1-1.8, 2.1-2.10, 3.1, 3.5, 3.6, and 4.2-4.7.

- **Basic functions (measurements)/units**
  - Must know basic functions and their graphs: linear functions, quadratic polynomials, $1/x$, e^x, $\ln x$, $\sin x$, $\cos x$.
  - Must know how to identify horizontal and vertical shifts of basic functions from the graphs and from their formulas. Similarly for vertical and horizontal stretchings.
  - Must know how to properly change units. And how that can affect measurements.
  - Must understand composition of functions and inverses when they exist.
- **Discrete Dynamical Systems**
  - For a general discrete dynamical system: $m_{n+1} = f(m_n)$, must understand the role of the *updating function* f, as connecting the system at time n to the system at time $n+1$ (one unit of time later).
    - Must be able to find m_n from the initial measurement m_0 by iterating the system n-times,
    $$m_0 \to m_1 = f(m_0) \to m_2 = f(m_1) \to \cdots \to m_n = f(m_{n-1}).$$
    - Must be able to do a graph of time vs measurement (n, m_n), to gain some feeling about the system.
  - Must understand that the system $b_{t+1} = rb_t$ has a closed solution $b_t = r^t b_0$ in terms of $r > 0$ and the initial state of the system b_0.
    - If $r > 1$ we have exponential growth, and must be able to find *doubling time*.
    - If $r < 1$ we have exponential decay, and must be able to find *half-life*.
  - Must be able to use *cob-webbing* to learn about qualitative long-time behaviour of the system.
    - Must understand what is an *equilibrium point* (a point that doesn't change as the system evolves, a fixed point), how to find them by solving the equation $f(x) = x$.
    - Must understand the concepts of *stable equilibrium points* (if the system starts nearby such equilibrium point it gets closer and closer to the point as time goes to infinity).

    * Must understand the concept of *unstable equilibrium points* (if the system starts no matter how close but not at such equilibrium point, it will drift away).
    * Must understand well discrete dynamical systems where the updating function is linear $f(x) = mx+b$ and slope test to determine whether an equilibrium point is stable ($m < 1$), or unstable ($m > 1$).
    * For general updating functions must understand the *slope/derivative test* to determine whether an equilibrium point x is stable ($f'(x) < 1$), or unstable ($f'(x) > 1$).

- **Limits and continuity**
  - Must be confortable with the statement: $\lim_{x\to a} f(x) = L$ means that as x gets closer to a, $f(x)$ gets closer to L.
  - Must be able to compute limits: numerically, using continuity (if f is continuous at a then $\lim_{x\to a} f(x) = f(a)$, so we can substitute), or using basic rules (sum, multiplication,quotient).
  - Must be able to identify when limits do NOT exist (limit from the right and from the left don't coincide, function is going to infinity, or oscillating wildly)
  - Must be able to identify when a function is NOT continuous: limit doesn't exit, or it exists but it does not coincide with $f(a)$.
  - Must be able to use L'Hopital's rule to calculate limits of indeterminate forms ($0/0$, ∞/∞).
  - Must be able to rank functions according to how fast they grow at infinity and use the limit test comparison to decide on a ranking. In particular should know that at infinity, logarithms grow slower than any positive power, and polinomials grow slower than exponentials:

$$\ln x << x^{\alpha} << e^{x}.$$

- **Differentiation**
  - Must understand interpretation as instantaneous rate of change: limit of average rates of change.
  - Must understand geometric interpretation of the derivative of a function f at x as the slope of the tangent line to the graph of f at the point $(x, f(x))$.
  - Must understand that the slope of the tangent line at $(x, f(x))$ is the limit of the slopes of the secant lines going through that point and another point on the graph which is getting closer to the given point.
  - Must be able to find the equation of the tangent line at $(x, f(x))$, and be familiar with the idea that this tangent line provides the best linear approximation to the function near, and at, the given point.
  - Must be able to identify when a function is NOT differentiable: if it is not continuous, or if it is when the derivative doesn't exist, for example if it has corners, or vertical tangent lines.

- **Calculus of derivatives**
  - Must know derivatives of basic functions: C, x, x^n, e^x, $\ln x$, $\sin x$, $\cos x$.
  - Must know basic rules of differentiation and how to use them to compute derivatives of more complicated functions. (sum rule, product rule, quotient rule, power rule, chain rule, inverse function rule).
- **Applications of derivatives**
  - Must know how to use information from first and second derivative to graph a function (e.g. $f(x) = e^{-x^2}$). Must be able:
    * to identify intervals of increase/decrease,
    * to identify maximums and minimums,
    * to identify intervals of convexity (up/down)
    * to identify inflexion points
  - Must be able to sketch the graph of the derivative of f given the graph of f.
  - Must understand that if the function analyzed is position, then its derivative is the velocity, and its second derivative is acceleration.
- **Integration**
  - Antiderivatives/indefinite integrals
    * Must understand that the antiderivative of a function f is another function F whose derivative is exactly f ($F' = f$). Once an antiderivative is found, a whole family is found: $F + C$, where C is a constant.
    * Must be able to find antiderivatives for simple functions: powers (x^n, n can be positive or negative, integer or not), exponentials (e^x), trigonometric functions ($\sin x$, $\cos x$).
    * Must be able to use the linear property of indefinite integrals to calculate antiderivatives of linear combinations of simple functions,
    $$\int \big(af(x) + bg(x)\big)\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$
    * Must be able to use integration by parts to calculate indefinite integrals for simple examples like: xe^x, $\ln x$, $e^x \sin x$.
    * Must be able to use the method of substitution to calculate indefinite integrals for simple examples like: $\tan x = \frac{\sin x}{\cos x}$, xe^{-x^2}.
    * Must be able to solve the simple differential equation: find f given its derivative f', and knowing that $f(x_0) = a$ for some point x_0.
  - Definite integrals
    * Must understand that the definite integral $\displaystyle\int_a^b f(t)\,dt$ is a number that can be calculated using the Fundamental Theorem of Calculus by $\big(F(b) - F(a)\big)$ where F is an antiderivative of f (that is $F' = f$).

  - Must understand that this number represents the signed area under the graph of the function f between $t = a$ and $t = b$.
  - Must be familiar with the idea that integrals are limits of sums (so-called *Rieman sums*), and be able to use data (given time intervals and the value of the function to be integrated at one point on each time interval) to approximate the value of the integral.
- Improper integrals
  - Must be able to calculate simple improper integrals with infinite limits of integration, and understand that sometimes this leads to finite areas (convergence), others it does not (divergence). E.g. $\int_1^\infty \frac{1}{x^n}\,dx$, for $n > 0$ (convergent if $n > 1$, divergent if $0 < n \leq 1$).
  - Must be able to calculate simple improper integrals with infinite integrands, and understand that sometimes this leads to finite areas (convergence), others it does not (divergence). E.g. $\int_0^1 \frac{1}{x^n}\,dx$, for $n > 0$ (convergent if $0 < n < 1$, divergent if $n \geq 1$).

Practice Problems for the final (previews review problems, as well as midterm problems are good practice too). We have plenty of problems for integration in the last homework, so I am not listing more problems for integration.

- Section 1.3: 31-33.
- Section 1.5: 43.
- Section 1.6: 47-50.
- Supplemental problems to Chapter 1: 12, 23.
- Section 2.4: 17, 21, 31-34.
- Section 2.7: 3, 8.
- Section 2.10: 27, 41-46.
- Supplemental problems to Chapter 2: 2, 5, 7, 10, 17, 18, 20, 26, 28, 30.
- Supplemental problems to Chapter 3: 1, 5 (write the equation of the tangent line to the given function at $(0, f(0))$, and evaluate the line at the point $x = -0.03$, this should give an approximation of the value of $f(-0.03)$, compare to the value the calculator will give), 7, 8.