MATH 180 - Review Midterm 1 - Fall 2008

p. 128-129 Suplementory Problems (1)
Instructor: C. Pereyra

(1) (p.128) density of each bacterium $\rho = 2.0\ g/cm^3$

(a) if size of each bacterium $= 5\,\mu m \times 5\,\mu m \times 20\,\mu m = V$
(volume)

Find number of bacteria $= N$
if Total mass $= 30\,gr = M$

Recall
$1\,\mu m = 10^{-6}\,m = 10^{-4}\,cm$
$10^2\,cm = 1\,m$

M = Total mass = Number of bacteria x mass of each bacterium

m = mass of one bacterium = density each bacterium X size each bacterium

so $M = N \times m$

$$m = \rho \times V = 2.0\,\frac{g}{cm^3} \times 500\,(\mu m)^3 \times \left(\frac{10^{-4}\,cm}{\mu m}\right)^3$$

$$m = 2.0 \times 5 \times 10^2 \times 10^{-12}\,\frac{g}{\cancel{cm^3}}\,\frac{\cancel{(\mu m)^3}\,\cancel{cm^3}}{\cancel{(\mu m)^3}}$$

$m = 10^{-9}\,g$ ← mass of one bacterium

$$N = \frac{M}{m} = \frac{30}{10^{-9}}\,\frac{g}{g} = \boxed{30 \times 10^9}$$ Total number of bacteria.

(b) Suppose Known: $300\,(\mu m)^3 \le V \le 750\,(\mu m)^3$

Then $2 \times 300 \times 10^{-12}\,g \le m \le 2 \times 750 \times 10^{-12}\,g$ (Using $m = \rho \times V$)

individual mass ranges → $6 \times 10^{-10} \le m \le 15 \times 10^{-10}\,g$

$N = M/m$ using

$$\frac{30}{15 \times 10^{-10}} \le N \le \frac{30}{6 \times 10^{-10}}$$

Total number of bacteria ranges between →

$$\boxed{2 \times 10^{10} \le N \le 5 \times 10^{10}}$$

$$2 \times 10^{10} \le N \le 5 \times 10^{10}$$

MATH 180 – Review Midterm 1 – Fall 2008

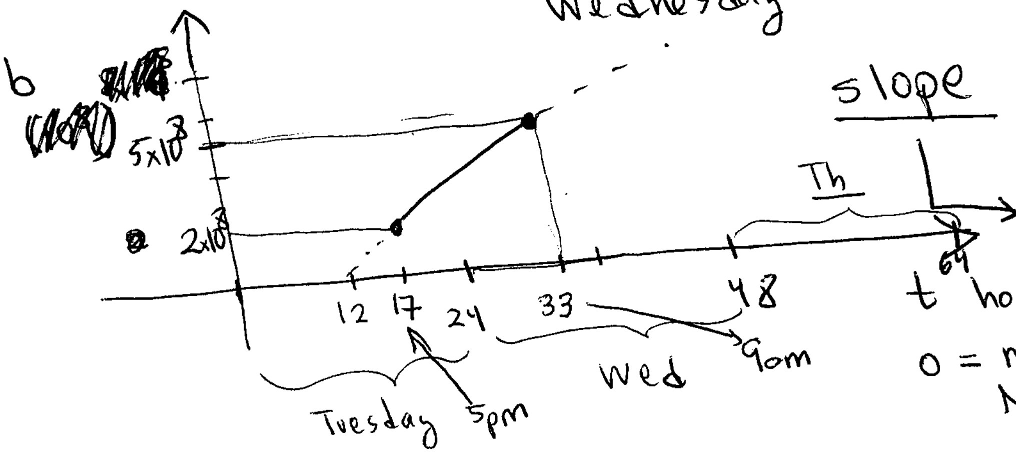
p. 128–129 Suplementary Problems (2)

(2) (p. 128) Suppose number of bacteria is a linear function of time

a) If at 5 pm on Tuesday $b(t) = 2\times10^8$

at 9 am on Wednesday $b(t) = 5\times10^8$

2 pts to determine equation of the line



0 = midnight from Monday to Tuesday

slope $m = \dfrac{5\times10^8 - 2\times10^8}{33-17}$

$$\boxed{m = \frac{3\times10^8}{16}}$$

$$(b - 2\times10^8) = m(t-17) = \frac{3\times10^8}{16}(t-17)$$

$$b = \frac{3\times10^8}{16}t - \frac{3\times17\times10^8}{16} + 2\times10^8$$

$$b = \frac{3\times10^8}{16}t + \left(2 - \frac{3\times17}{16}\right)\times10^8$$

$$\boxed{b = \frac{3\times10^8}{16}t - \frac{19}{16}\times10^8}$$

(b) Find t so that $b(t) = 1.1\times10^{+9}$

solve $1.1\times10^9 = \left(\frac{3}{16}t - \frac{19}{16}\right)\times\cancel{10^8}$

$$11\times16 = 3t - 19$$

$$176 + 19 = 3t$$

$$195 = 3t \longrightarrow \boxed{t = 65 \text{ hours}}$$

$\boxed{\text{at 11 pm on Thursday}}$

2 (Continuation p. 128)

(c) Lab across the hall has another culture st

at 5pm on Tuesday $B_t = 2\times 10^8$ $(t=17)$

9am on Wednesday $B_t = 3.4\times 10^8$ $(t=33)$

Their linear equation has slope

$$M = \frac{3.4\times 10^8 - 2\times 10^8}{33-7} = \frac{1.4\times 10^8}{16}$$

And their equation is

$$B - 2\times 10^8 = M(t-17)$$

$$B = \frac{1.4}{16}\times 10^8\, t - \frac{1.4\times 17}{16}\times 10^8 + 2\times 10^8$$

$$B = \left[\frac{1.4}{16}t + \left(2 - \frac{1.4\times 17}{16}\right)\right]\times 10^8$$

Our culture will have twice as many bacteria as theirs when: (we must solve for t)

$$b(t) = 2B(t)$$

$$\left(\frac{3}{16}t - \frac{19}{16}\right)\times \cancel{10^8} = 2\left[\frac{1.4}{16}t + \frac{(32 - 1.4\times 17)}{16}\right]\cancel{\times 10^8}$$

$$= \frac{2.8}{16}t + \frac{64 - 2.8\times 1.7}{16}$$

$$\frac{0.2}{\cancel{16}}t = \frac{64 - 2.8\times 17 + 19}{\cancel{16}} = 83 - 47.6$$

$$t = \frac{35.4}{0.2} = \boxed{177 \text{ hours} = t}$$

$t = 177$ hours

$\frac{177}{24} = 6 + \frac{2}{3}$

6 days and 16 hours

Next Monday at 4pm

We will have twice as many at $\boxed{4 \text{ pm on Monday next week}}$

③ (p.128) Consider $f(x) = e^{-2x}$, $g(x) = x^3 + 1$

(a) Find inverses

f: $0 < y = e^{-2x}$

$\ln y = -2x$

$x = -\frac{1}{2}\ln y$

$x = \ln\left(y^{-1/2}\right)$ $y > 0$

$x = \ln\left(\frac{1}{\sqrt{y}}\right)$

$$\boxed{f^{-1}(y) = \ln\left(\frac{1}{\sqrt{y}}\right)}$$

g: $y = x^3 + 1$

$y - 1 = x^3$

$\sqrt[3]{y-1} = x$

$$\boxed{g^{-1}(y) = \sqrt[3]{y-1}}$$

$f(x) = 2 \Rightarrow$ $\boxed{x = f^{-1}(2) = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2}$

$g(x) = 2 \Rightarrow$ $\boxed{x = g^{-1}(2) = \sqrt[3]{2-1} = \sqrt[3]{1} = 1}$

(b) $\boxed{f \circ g(x)} = f(g(x)) = e^{-2(x^3+1)} = \boxed{e^{-2x^3-2}}$

$\boxed{g \circ f(x)} = g(f(x)) = \left(e^{-2x}\right)^3 + 1 = \boxed{e^{-6x} + 1}$

$f \circ g(2) = e^{-2 \times 2^3 - 2} = \boxed{e^{-18} = f \circ g(2)}$

$g \circ f(2) = e^{-6 \times 2} + 1 = \boxed{e^{-12} + 1 = g \circ f(2)}$

(3) (p128 continuation)

(c) Find inverse $g \circ f(x) = e^{-6x} + 1$

$$y = e^{-6x} + 1 \geq 1$$

(solve for x) $y - 1 = e^{-6x}$

$$\ln(y-1) = -6x \longrightarrow x = -\frac{1}{6}\ln(y-1)$$

$y > 1$

$$x = \ln(y-1)^{-1/6}$$

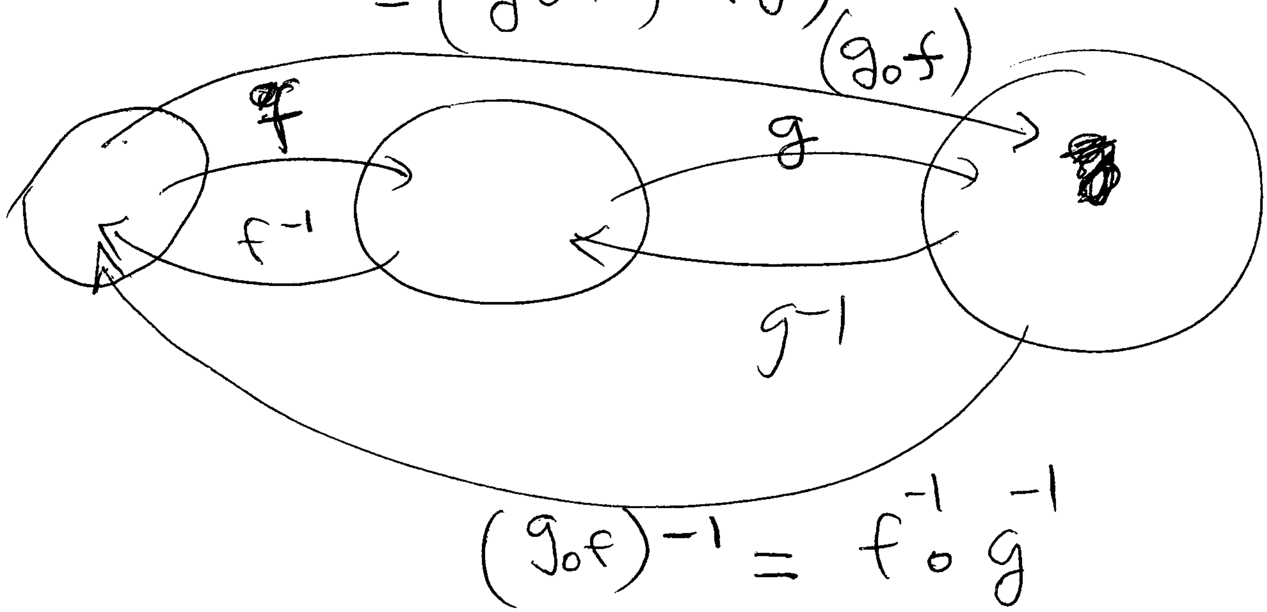
so $\boxed{(g \circ f)^{-1}(y) = -\frac{1}{6}\ln(y-1), \quad \underbrace{y > 1}_{\text{domain}}}$

Note that

$$(f^{-1} \circ g^{-1})(y) = f^{-1}(g^{-1}(y)) = -\frac{1}{2}\ln(g^{-1}(y))$$

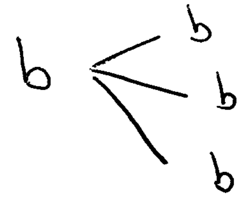
$$= -\frac{1}{2}\ln(y-1)^{1/3} = -\frac{1}{6}\ln(y-1)$$

$$= (g \circ f)^{-1}(y)$$



$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

(4) (p. 128)



each bacteria splits in 3 after 2 hours
Nobody dies
Dynamical system is:

(★) unit of time 2 hours	$b_{t+1} = 3 b_t$ (two hours later)

(a) updating function: $\boxed{f(x) = 3x}$

(b) $t=0 \quad b_0 = 2 \times 10^7 \qquad (t=0 \quad 9\text{ AM})$

$t=4 \quad b_4 = \,?\qquad (t=4 \quad 5\text{ pM})$

There are 8 hours in between, That is 4 units of time

Notice That The solution of (★) is

$$b_t = 3^t b_0$$

So $b_4 = 3^4 \times 2 \times 10^7 = \boxed{162 \times 10^7}$

at 5pm There are This many bacteria

(c) $\boxed{b_t = 3^t b_0} = 3^t \times 2 \times 10^7$

(d) Find t so That $b_t = 10^9 = 3^t \times b_0$

assuming $b_0 = 2 \times 10^7$, solve for t

$$10^9 = 3^t \times 2 \times 10^7$$

$$\frac{100}{2} = 3^t$$

$$3^t = 50$$

$$e^{t \ln 3} = 50$$

$$\boxed{t = \frac{\ln 50}{\ln 3} = \log_3 50}$$

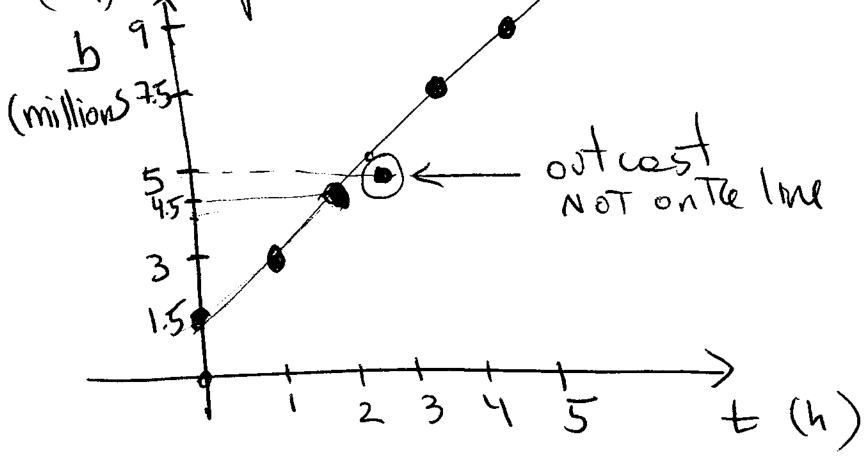
blocks of 2 hours

\# hours after experiment began $= 2t$

(5) (p. 128)

Time t (h)	Number, b_t (millions)
0.0 h	1.5
1.0 h	3.0
2.0 h	4.5
3.0 h	5.0
4.0 h	7.5
5.0 h	9.0

(a) Graph



(b) Line: $m = \dfrac{3.0-1.5}{1.0-0.0} = \dfrac{1.5}{1} = \dfrac{3}{2}$ slope

b-intercept is $1.5 = b_0$

$b = mt + b_0$ $\boxed{b = 1.5t + 1.5}$

(c) The point $t = 3$ h, $b_t = 5$ million is not on the line

~~[illegible]~~ $t = 3$ $b = 1.5 \times 3 + 1.5 = 6$

(d) At time $t = 7$ hours, I would expect

$b = 1.5 \times 7 + 1.5 = \boxed{12 \text{ million}}$

(6) (p 128) [I did it in class last thursday]

(7), (8) (p128) [Trigonometric functions will not be evaluated this time].

(9) (p 128-129) [I did it in class last Thursday]

Instead I e-mailed you 13 & 15

(10) I forgot and it is in p. 11

(13) (p. 129) $A_{t+1} = A_t + .1\, A_t = (1.1)\, A_t$

(Area)

at 2pm $\to t=0$ $\boxed{A_0 = 2\,cm^2}$ $\boxed{t \text{ in hours}}$

(a) Area at 5pm? That is at $t=3$

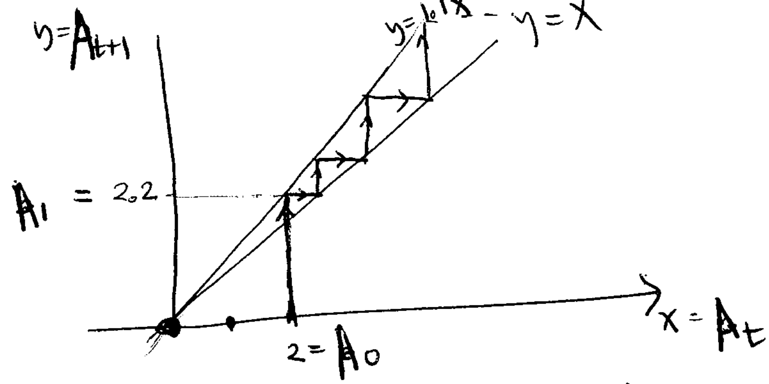
$2 = A_0 \longrightarrow A_1 = 1.1 \times 2 = 2.2\,cm^2 \xrightarrow{B} A_2 = 1.1 \times 2.2\,cm^2$

$A_2 = 2.42\,cm^2 \longrightarrow A_3 = (1.1)\times(2.42)\,cm^2$

$\boxed{A_3 = 2.662\,cm^2}$

(b) Discrete dynamical system: $\boxed{A_{t+1} = (1.1)\,A_t}$

updating fcn $f(x) = 1.1\,x$



(c) Area at 1pm: $A_t = \frac{1}{1.1} A_{t+1}$

at 1 pm That is at $t=-1$ starting from $b_0 = 2$.

So $A_{-1} = \frac{1}{1.1} A_0 = \frac{2}{1.1} = 1.\overline{81} \sim 1.82$

$\boxed{\text{At 1 pm The area is } 1.\overline{81}\,cm^2}$

(d) Assume all bacteria have same size, (Area) each adult produces two off-springs each hour

$b \xrightarrow{} b$, $b \xrightarrow{2\text{ hours}} b$

~~How many~~ what fraction survives?

(13) (d) (p. 129 continuation)

Let b_t = Number of bacteria at time t each one of area a

Total area at time t is

$$A_t = a \times b_t$$

We know dynamical systems for the area:

$$A_{t+1} = 1.1\, A_t$$

we know $b_{t+1} = 2 \times \alpha \times b_t$

α ← fraction that survives.

Must solve for α

$$A_{t+1} = a\, b_{t+1}$$
$$A_t = a\, b_t$$
$$A_{t+1} = 1.1\, A_t \Rightarrow \cancel{a} b_{t+1} = 1.1\, \cancel{a}\, b_t$$

$$\boxed{b_{t+1} = 1.1\, b_t}$$

so $2\alpha = 1.1$

$$\boxed{\alpha = \frac{1.1}{2} = 0.55}$$

← Fraction that survives is $\frac{55}{100}$, a little bit more than half.

(e) Find t so that $A_t = 10\ cm^2$

$$A_t = (1.1)^t A_0 = (1.1)^t\, 2\ cm^2$$

solve $10 = (1.1)^t \times 2 \rightarrow e^{t \ln(1.1)} = 5 \rightarrow \boxed{t = \frac{\ln 5}{\ln 1.1}}$

15 (p. 129) b_t = bacterial population at a given generation

b_{t+1} = bacterial population next generation

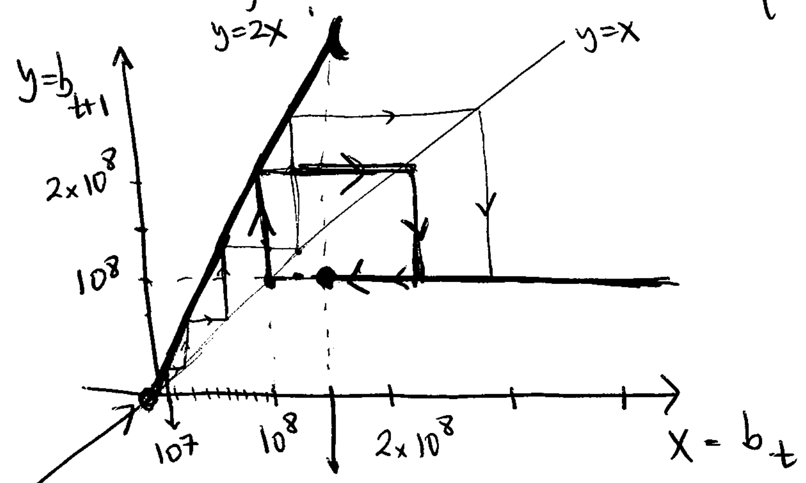
units of time = generations.

Dynamical system:

$$b_{t+1} = \begin{cases} 2b_t & \text{if } b_t < 1.5 \times 10^8 \\ 1.0 \times 10^8 & \text{if } b_t \geq 1.5 \times 10^8 \end{cases}$$

Updating fcn: $f(x) = \begin{cases} 2x & \text{if } x < 1.5 \times 10^8 \\ 10^8 & \text{if } x \geq 1.5 \times 10^8 \end{cases}$

(a)



equilibrium (unstable)

(b) $b_0 = 10^7$

$b_1 = 2 \times 10^7$

$b_2 = 4 \times 10^7$

$b_3 = 8 \times 10^7 < \frac{3}{2} \times 10^8$

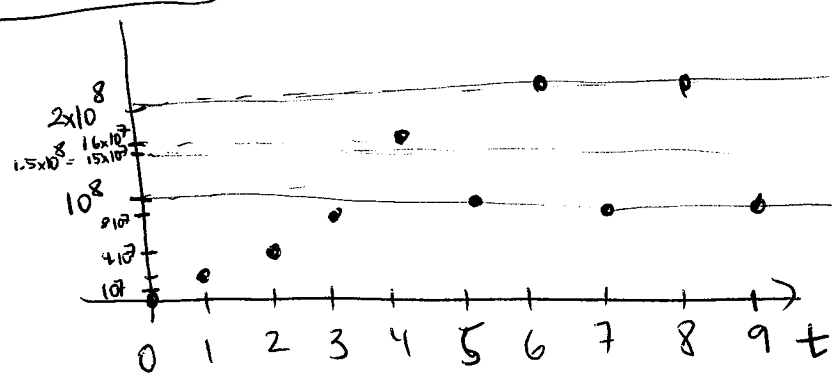
increases exponentially until $> 15 \times 10^7$ $b_4 = 16 \times 10^7 > 15 \times 10^7$

Then oscillates between 10^8 and 2×10^8

$b_5 = 10^8 < 1.5 \times 10^8$

$b_6 = 2 \cdot 10^8 > 1.5 \times 10^8$

$b_7 = 10^8 <$



(c) Only one equilibrium point $t=0$ and is unstable.

(10) (p. 129) Total mass $M = 3 \times 10^{-3}$ g

each bacteria's mass $m = 2 \times 10^{-10}$ gr

density $d = 1.5\ g/cm^3 = \dfrac{\text{mass}}{\text{volume}}$

each bacteria is spherical so each bacteria's volume $\boxed{V = \frac{4}{3}\pi r^3}$ ☆ where r is the radius.

(a) $N = ?$ where N = total number of bacteria.

Total mass = Number bacteria x individual mass

$$M = \underset{\text{unknown}}{N} \times m \rightarrow N = \frac{M}{m} = \frac{3 \times 10^{-3}}{2 \times 10^{-10}}$$

$$\boxed{N = \frac{3}{2} 10^{7} \text{ bacteria}}$$

(b) Solve for r in formula for volume sphere ☆

Need volume of a bacteria, get it from it mass and the density:

$$d = \frac{m}{V} \Rightarrow V = \frac{m}{d} = \frac{2 \times 10^{-10} g}{1.5\ g/cm^3}$$

$$\Rightarrow \boxed{V = \frac{2}{1.5} 10^{-10} cm^3} \quad \text{but } V = \frac{4}{3}\pi r^3$$

$$\text{so } r^3 = \frac{3}{4}\frac{V}{\pi} = \underbrace{\frac{3}{4} \cdot \frac{2}{3/2}}_{=1} \frac{10^{-10}}{\pi} cm^3 = \frac{10^{-10}}{\pi} cm^3$$

$$\boxed{r = \sqrt[3]{\frac{10^{-10}}{\pi}}\ cm} \quad \text{radius}$$

(c) $d = \dfrac{M}{V} \rightarrow$ Total $V = \dfrac{M}{d} = \dfrac{3 \times 10^{-3} g}{1.5\ g/cm^3} = \boxed{2 \times 10^{-3} cm^3}$ Total volume