Moduli Problems in Sasakian Geometry

Charles Boyer

University of New Mexico

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- We give partial answers to these problems for particular cases. My talk is based on joint work with various colleagues: Leonardo Macarini, Justin Pati, Christina Tønnesen-Friedman, and Otto van Koert.

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- A **contact 1-form** η such that

$$\eta \wedge (d\eta)^n \neq 0.$$

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for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So $\{$oriented contact 1-forms in $\mathcal{D}\} \approx C^\infty(M)^+$

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- The pair $(\mathcal{D},J)$ is a **strictly pseudo-convex almost CR structure** (sψCR structure).
- If $(\mathcal{D},J)$ is an **integrable** CR structure, and $\mathcal{L}_\xi g=0$ then $\mathcal{S}=(\xi,\eta,\Phi,g)$ is a **Sasakian** structure. Then contact manifold $(M,\mathcal{D})$ is of **Sasaki type**.

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- Under the right assumptions $SH^{+,S^1}(W)$ is a **contact invariant**.

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For a convenient Liouville filling (W, ω), the **mean Euler characteristic** is defined by

$$\chi_m(W) = \frac{1}{2}\left(\liminf_{N\to\infty} \frac{1}{N}\sum_{i=-N}^{N}(-1)^i sb_i(W) + \limsup_{N\to\infty} \frac{1}{N}\sum_{i=-N}^{N}(-1)^i sb_i(W)\right)$$

if this number exists.

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if this number exists.

- Under various technical assumptions, $\chi_m(W)$ exists and is a **contact invariant** independent of the Liouville filling.

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- Assume the filling is **Liouville**, define the **symplectic Betti numbers** by $sb_i := \mathrm{rank}\ SH_i^{+,S^1}(W)$.

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- Under various technical assumptions, $\chi_m(W)$ exists and is a **contact invariant** independent of the Liouville filling.
- $\chi_m(W)$ and $SH^{+,S^1}(W)$ allows us to distinguish components of the Sasaki moduli space.

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Brieskorn Manifolds - Rational Homology Spheres, B-, Macarini, van Koert

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- I present two **fundamental theorems** about $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ and then present brief outlines of their proofs. Finally, I discuss the special case of S^3-bundles over a Riemann surface Σ_g.

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- Relation to **CR Yamabe Problem** (Jerison and Lee): For a Sasaki structure the Webster pseudo-Hermitian metric coincides with the transverse Kähler metric. So a **CSC** Sasaki metric provides a solution to the CR Yamabe Problem. It is know that when the **CR Yamabe invariant** $\lambda(M)$ is **nonpositive**, the CSC metric is unique. However, when $\lambda(M) > 0$ there can be several CSC solutions. Our results provides many such examples.

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- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre).
- When $c_1(\mathcal{D}) = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram on the manifold $S^2 \times S^3$.
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- Similar results hold for the non-trivial S^3-bundle over S^2, but no SE metrics.

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- When $l_2 > 1$ some of the same type of results have been obtained on 5-manifolds whose fundamental group is a non-Abelian extension of $\pi_1(\Sigma_g)$ in Castañeda's thesis.

THANK YOU!

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