Extremal Sasakian Metrics on S^3-bundles over Riemann Surfaces

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Joint work with Christina Tønnesen-Friedman

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- Given a manifold determine how many contact structures of Sasaki type there are.

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- The pair $(\mathcal{D}, J)$ is a **strictly pseudo-convex almost CR structure** (sψCR structure).

Sasakian manifold

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- Sasakian structure gives pseudo convex CR structure $(\mathcal{D}, J)$ and a transverse holomorphic structure $(\xi, \bar{J})$. The former fixes the contact structure while the latter fixes the characteristic foliation.

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- **maximal torus** with $0 \leq k \leq n+1$

$$\begin{array}{ccccc} & \nearrow & \mathfrak{CR}(\mathcal{D},J) & \searrow & \\ T^k \subset \mathfrak{Aut}(\mathcal{S}) & & & & \mathfrak{Con}(M,\mathcal{D}). \\ & \searrow & \mathfrak{Con}(M,\eta) & \nearrow & \end{array}$$

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- The distinct Sasaki cones $\kappa(\mathcal{D}, J_\alpha)$'s correspond to distinct **conjugacy classes of maximal tori** in $\mathfrak{Con}(M, \mathcal{D})$.

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- Special case: constant scalar curvature Sasakian (cscS). If $c_1(\mathcal{D}) = 0 \Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta, \ a, b$ constants. If $b = 0$ get Sasaki-Einstein (SE).

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Extremal Sasakian metrics

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^\# s_g$ is transversely holomorphic.
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Theorem

An extremal Sasaki metric g has constant scalar curvature if and only if $\mathfrak{F} = 0$.

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Theorem (Futaki,Ono,Wang,Cho)

Every toric contact structure of Reeb type with $c_1(\mathcal{D}) = 0$ admits a unique Sasaki-Einstein metric

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S^3-bundles over Σ_g (B-,Tønnesen-Friedman)

- Non-toric case $g \geq 1$: maximal torus has dimension 2.
- Join Construction: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ (B-,Galicki,Ornea)
- $M_1 \star_{l_1,l_2} M_2$ has a natural Sasakian structures for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
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- The proof uses a recent topological rigidity argument of Kreck,Lück.

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- Similar results hold for the non-trivial S^3-bundle over Σ_g (still in progress).

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