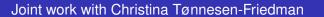
Extremal Sasakian Metrics on S³-bundles over Riemann Surfaces Santiago de Compostela, Conference in Honor of Luis Cordero

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- Given a manifold determine how many contact structures of Sasaki type there are.

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• The pair (\mathfrak{D}, J) is a **strictly pseudo-convex almost CR structure** (s ψ CR structure).

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The contact metric structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ is **K-contact** if $\mathcal{L}_{\xi}g=0$ (or $\mathcal{L}_{\xi}\Phi=0$). It is **Sasakian** if in addition (\mathcal{D},J) is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler. In the latter case we say that the contact structure \mathcal{D} is of **Sasaki** type.

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- Sasakian structure gives pseudo convex CR structure (\mathfrak{D},J) and a transverse holomorphic structure (ξ,\bar{J}) . The former fixes the contact structure while the latter fixes the characteristic foliation.

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Sasaki cones and bouquets (B-Galicki-Simanca,B-)

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- The distinct Sasaki cones $\kappa(\mathcal{D}, J_{\alpha})$'s correspond to distinct conjugacy classes of maximal tori in $\mathfrak{Con}(M, \mathcal{D})$.

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Theorem

An extremal Sasaki metric g has constant scalar curvature if and only if $\mathfrak{F} = 0$.

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- Special Case: Toric Sasakian structures on S³-bundles over S².

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 $Y^{p',q'}$ is isomorphic to $Y^{p,q}$ if and only if p'=p. So there is a $\phi(p)$ -bouquet of Sasaki cones on $Y^{p,q}$ and there are $\phi(p)$ Sasaki-Einstein metrics where $\phi(p)$ is the Euler phi function.

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• The proof uses a recent topological rigidity argument of Kreck, Lück.

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• For any genus $g \ge 1$ and for each positive integer k, the contact manifold $(\Sigma_g \times S^3, \mathcal{D}_k)$ has a 2-dimensional Sasaki cone $\kappa(\mathfrak{D}_k, J_0)$ with a unique regular ray of CSC Sasakian structures.

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- Similar results hold for the non-trivial S^3 -bundle over Σ_g (still in progress).

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