The Join Construction & Extremal Sasakian Geometry

AMS Special Session on Manifolds with Special Holonomy and Generalized Geometries

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Joint work with Christina Tønnesen-Friedman

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- The pair $(\mathcal{D}, J)$ is a **strictly pseudo-convex almost CR structure** (sψCR structure).

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- **Transverse homothety**: If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a Sasakian structure, so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for every $a \in \mathbb{R}^+$ with $g_a = ag + (a^2 - a)\eta \otimes \eta$. So Sasakian structures come in rays.

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- Determine those of **constant scalar curvature (CSC)**.

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- **maximal torus** with $0 \le k \le n+1$

$$\begin{array}{ccccc} & & \mathfrak{CR}(\mathcal{D},J) & & \\ & \nearrow & & \searrow & \\ T^k \subset \mathfrak{Aut}(\mathcal{S}) & & & & \mathfrak{Con}(M,\mathcal{D}). \\ & \searrow & & \nearrow & \\ & & \mathfrak{Con}(M,\eta) & & \end{array}$$

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- The distinct Sasaki cones $\kappa(\mathcal{D},J_\alpha)$'s correspond to distinct **conjugacy classes of tori** in $\mathfrak{Con}(M,\mathcal{D})$. They are distinguished by **equivariant Gromov-Witten invariants**.

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- The Sasaki-Futaki invariant $\mathfrak{F}(X) = \int_M X(\psi_g) d\mu_g$ where X is transversely holomorphic and ψ_g is the Ricci potential satisfying $\rho^T = \rho_h^T + i\partial\bar{\partial}\psi_g$ where ρ^T is the transverse Ricci form and ρ_h^T is its harmonic part. An extremal Sasaki metric g has constant scalar curvature if and only if $\mathfrak{F} = 0$.

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- Further analysis of the 4th order polynomial proves the remaining statements of the theorem.

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- For $n > 0$, the work of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms shows that extremality boils down to: the transverse Kähler structure is $g_{\mathcal{D}} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = \omega_{\Sigma_g}$ the standard area form, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{q}, \Theta'(1) = -\frac{2}{p}$. When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})$ is a 4th order polynomial we get extremal Kähler transverse metrics; hence, extremal Sasaki metrics.
- Demanding a 3rd order polynomial one shows that each Sasaki cone has a unique **CSC** structure.
- The distinct Sasaki cones in the bouquet $\mathfrak{B}_k$ correspond to distinct conjugacy classes of maximal tori in $\mathfrak{Con}(\mathcal{D}_k)$. The classes corresponding to $m = 0, \cdots, k-1$ are shown to be distinct using the work of Buşe on equivariant Gromov-Witten invariants.
- Further analysis of the 4th order polynomial proves the remaining statements of the theorem.
- □

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- Some of the same type of results can be obtained on 5-manifolds whose fundamental group is a non-Abelian extension of $\pi_1(\Sigma_g)$.

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- For each odd dimension ≥ 3 there exists a countable infinity of Sasakian manifolds with a perfect fundamentfinial group which admit CSC Sasaki metrics. Furthermore, there is an infinite number of such Sasakian manifolds that have the integral cohomology ring of $S^2 \times S^{2r+1}$.

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- There exist a countably infinite number of aspherical contact 5-manifolds with perfect fundamental group and the integral cohomology ring of $S^2 \times S^3$ that admit CSC Sasaki metrics. Moreover, there are such manifolds that admit a ray of Sasaki-η-Einstein metrics (hence, Lorentzian Sasaki-Einstein metrics).

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