Extremal Sasakian Metrics

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March 2012

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- Determine the (pre)-moduli space of extremal Sasakian structures
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- Given a manifold determine how many contact structures of Sasaki type there are.

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• The pair (\mathfrak{D}, J) is a **strictly pseudo-convex almost CR structure** (s ψ CR structure).

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The contact metric structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ is **K-contact** if $\mathcal{L}_{\xi}g=0$ (or $\mathcal{L}_{\xi}\Phi=0$). It is **Sasakian** if in addition (\mathcal{D},J) is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler.

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- Nested structures: Sasakian ⊂ sψCR ⊂ Contact
- Sasakian structure gives pseudo convex CR structure (\mathfrak{D}, J) and a transverse holomorphic structure (ξ, \bar{J}) . The former fixes the contact structure while the latter fixes the characteristic foliation.

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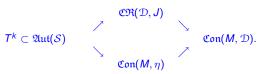
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• maximal torus with $0 \le k \le n+1$



Sasaki cones and bouquets (B-Galicki-Simanca,B-)

• Given a contact structure ${\mathfrak D}$ of Sasaki type

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- The Sasaki cones $\mathfrak{t}_{k(\alpha)}^+$'s correspond to the conjugacy classes of maximal tori in $\mathfrak{Con}(M, \mathbb{D})$.

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Theorem

An extremal Sasaki metric g has constant scalar curvature if and only if $\mathfrak{F}=0$.

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• Question: When is $\mathfrak{e}(\mathfrak{D}, J) = \mathfrak{t}_k^+(\mathfrak{D}, J)$?

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- $\mathfrak{e}(\mathfrak{D},J)$ is conical in the sense that if $S \in \mathfrak{e}(\mathfrak{D},J)$ so is S_a for all a>0. Moreover,

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Theorem (Futaki, Ono, Wang, Cho)

Every toric contact structure of Reeb type with $c_1(\mathfrak{D}) = 0$ admits a unique Sasaki-Einstein metric

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Theorem (B-Pati)

 $Y^{p',q'}$ is isomorphic to $Y^{p,q}$ if and only if p'=p. So there is a $\phi(p)$ -bouquet of Sasaki cones on $Y^{p,q}$ and there are $\phi(p)$ Sasaki-Einstein metrics where $\phi(p)$ is the Euler phi function.

S^3 -bundles over S^2 continued

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- Generally, we don't know how much of the Sasaki cones is represented by extremal Sasakian structures.

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Theorem (B-,Tønnesen-Friedman)

The manifold $M_g^3 \star_{k,1} S^3$ is diffeomorphic to $\Sigma_g \times S^3$ for each $k \in \mathbb{Z}^+$ and the contact structures \mathfrak{D}_k are all inequivalent. Moreover, there is a k-bouquet \mathfrak{B}_k of Sasakian structures and at least k conjugacy classes of maximal tori (circles) in $\mathfrak{Con}(\Sigma_g \times S^3, \mathfrak{D}_k)$.

The proof of the first statement uses a recent topological rigidity argument of Kreck, Lück. The
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- The genus g > 1 case is still in progress, so we focus on the genus one case.

Extremal Sasakian metrics in genus one case: $T^2 \times S^3$

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 The proof of this theorem uses the previous theorem together with the recent work of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on extremal Kähler metrics on ruled surfaces.

• One represents complex ruled surfaces as a projectivized rank two complex vector bundle $\mathbb{P}(\mathbb{1} \oplus L_{2n})$ over \mathcal{T}^2 where $\mathbb{1}$ is the trivial line bundle and L_{2n} is the line bundle of degree 2n.

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- The generic irregular Sasakian structure is handled by applying the openness theorem for extremal Sasaki metrics since quasi-regular Sasaki structures are dense in the Sasaki cone.
- The moduli space of extremal Sasakian structures is inherited from the moduli space of complex structures on the base ruled (orbifold) surfaces and they are typically non-Hausdorff.

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- Then consider the join $M^3 \star_{1,l} S^3$. If $L(a_0, \dots, a_n) \neq L(2, 3, 5)$ then $M^3 \star_{1,l} S^3$ has a perfect and infinite fundamental group and the homology of $S^2 \times S^3$.

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- Moreover, these 5-manifolds $M^3 \star_{1/1} S^3$ have natural Sasakian structures.

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