Extremal Sasakian Metrics

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- Given a manifold determine how many contact structures of Sasaki type there are.

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- The pair $(\mathcal{D}, J)$ is a **strictly pseudo-convex almost CR structure** (sψCR structure).

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- Sasakian structure gives pseudo convex CR structure $(\mathcal{D}, J)$ and a transverse holomorphic structure $(\xi, \bar{J})$. The former fixes the contact structure while the latter fixes the characteristic foliation.

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- **maximal torus** with $0 \leq k \leq n+1$

$$\begin{array}{ccccc} & & \mathfrak{CR}(\mathcal{D},J) & & \\ & \nearrow & & \searrow & \\ T^k \subset \mathfrak{Aut}(\mathcal{S}) & & & & \mathfrak{Con}(M,\mathcal{D}). \\ & \searrow & & \nearrow & \\ & & \mathfrak{Con}(M,\eta) & & \end{array}$$

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- The Sasaki cones $\mathfrak{t}_{k(\alpha)}^+$'s correspond to the conjugacy classes of maximal tori in $\mathfrak{Con}(M, \mathcal{D})$.

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- Calabi: Critical points have maximal symmetry.

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Extremal Sasakian metrics

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g$,
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Theorem

An extremal Sasaki metric g has constant scalar curvature if and only if $\mathfrak{F} = 0$.

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Theorem (Futaki,Ono,Wang,Cho)

Every toric contact structure of Reeb type with $c_1(\mathcal{D}) = 0$ admits a unique Sasaki-Einstein metric

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- 5-manifolds with perfect fundamental group and the homology of $S^2 \times S^3$.

S^3-bundles over S^2 (B-;B-Pati;Legendre)

- Exactly two S^3-bundles over S^2 determined by Stiefel-Whitney class $w_2(M^5) \equiv c_1(\mathcal{D}_{\mathbf{p}})$ *mod* 2: the trivial bundle $M^5 = S^2 \times S^3$ if $c_1(\mathcal{D}_{\mathbf{p}})$ is even, and the nontrivial X_∞ if $c_1(\mathcal{D}_{\mathbf{p}})$ is odd.

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S^3-bundles over S^2 continued

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- Generally, we don't know how much of the Sasaki cones is represented by extremal Sasakian structures.

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- The genus $g > 1$ case is still in progress, so we focus on the genus one case.

The genus one case: $T^2 \times S^3$ (B-,Tønnesen-Friedman)

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On $T^2 \times S^3$ there is a countable infinity of inequivalent contact structures $(M^3_1 \star_{k,1} S^3, \mathcal{D}_k)$ each with a k-bouquet of Sasaki cones of dimension two and each cone is exhausted by extremal Sasaki metrics. Moreover, each of these Sasaki cones has exactly one ray of constant scalar curvature metrics. Furthermore, there is a one dimensional cone (ray) of Sasakian structures that admits no extremal Sasaki metric. These structures all occur with moduli.

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- The proof of this theorem uses the previous theorem together with the recent work of Apostolov,Calderbank,Gauduchon,Tønnesen-Friedman on extremal Kähler metrics on ruled surfaces.

Extremal Sasakian metrics on $T^2 \times S^3$ continued

- One represents complex ruled surfaces as a projectivized rank two complex vector bundle $\mathbb{P}(1\!\!1 \oplus L_{2n})$ over T^2 where $1\!\!1$ is the trivial line bundle and L_{2n} is the line bundle of degree $2n$.

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- The moduli space of extremal Sasakian structures is inherited from the moduli space of complex structures on the base ruled (orbifold) surfaces and they are typically non-Hausdorff.

Work in Progress with C. Tønnesen-Friedman

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- Moreover, these 5-manifolds $M^3 \star_{1,l} S^3$ have natural Sasakian structures.

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