Maximal Tori in Contactomorphism Groups and Extremal Metrics

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• **Contact Manifold** (compact). *A* **contact 1-form** η *such that*

$$\eta \wedge (d\eta)^n \neq 0.$$

defines a **contact structure**

$$\eta' \sim \eta \iff \eta' = f\eta$$

for some $f \neq 0$, *take* $f > 0$. *or equivalently a codimension 1 sub-bundle* $\mathcal{D} = \mathrm{Ker}\ \eta$ *of* TM.
$(\mathcal{D}, d\eta)$ *symplectic vector bundle*

Unique vector field ξ, *called the* **Reeb vector field**, *satisfying*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

The **characteristic foliation** $\mathcal{F}_\xi$ *each leaf of* $\mathcal{F}_\xi$ *passes through any nbd* U *at most* k *times* $\iff$ **quasi-regular**, $k = 1 \leftrightarrow$ *regular, otherwise* **irregular**

Quasi-regularity is strong, most contact 1-forms are irregular.

• Contactomorphism Group

$\mathfrak{Con}(M, \mathcal{D}) =$

$\{\phi \in \mathfrak{Diff}(M) \mid \phi_* \mathcal{D} \subset \mathcal{D}\}$.

Alternatively,

$= \{\phi \in \mathfrak{Diff}(M) \mid \phi^* \eta = f(\phi, \eta)\eta\}$

with $f(\phi, \eta) > 0$ *everywhere.*

•$\mathfrak{Con}(M, \mathcal{D})$ *is a (regular) Fréchet Lie group (Lyčagin)*

A subgroup: $\mathfrak{Con}(M, \eta) =$

$\{\phi \in \mathfrak{Con}(M, \mathcal{D}) \mid \phi^* \eta = \eta\}$

• $\mathfrak{Con}(M, \eta)$ *is a closed Fréchet Lie subgroup of* $\mathfrak{Con}(M, \mathcal{D})$. *(B-)*

•**Important:** *Exp map is a local diffeomorphism for both groups. Recall* $Exp : \mathfrak{g} \to G$.

• **Maximal Tori** *in* $\mathfrak{Con}(M, \mathcal{D})$

• *Notation:* $\mathfrak{S}\mathfrak{C}_T(\mathcal{D}) = \emptyset \cup$ $\{$*conjugacy classes of maximal tori in* $\mathfrak{Con}(M, \mathcal{D})\}$

Decomposition:

$\mathfrak{S}\mathfrak{C}_T(\mathcal{D}) = \sqcup_{\mathfrak{r}=0}^{n+1} \mathfrak{S}\mathfrak{C}_T(\mathcal{D}, \mathfrak{r})$

$\mathfrak{r} =$ *dimension of maximal tori.*

$\mathfrak{n}(\mathcal{D}) =$ *cardinality of* $\mathfrak{S}\mathfrak{C}_T(\mathcal{D})$.

$\mathfrak{n}(\mathcal{D}, \mathfrak{r}) =$ *cardinality of* $\mathfrak{S}\mathfrak{C}_T(\mathcal{D}, \mathfrak{r})$.

A Torus $T^{\mathfrak{r}}$ is of *Reeb type* *if* $\exists$ *a contact form* η *s.t.* $\xi \in \mathfrak{t}_{\mathfrak{r}}$ *Lie algebra. Then* $T^{\mathfrak{r}} \subset \mathfrak{Con}(M, \eta)$

Contact bundle $\mathcal{D} \longrightarrow$ *choose* **almost complex structure** J *extend to* Φ *with* $\Phi\xi = 0$
with a compatible metric

$$g = d\eta \circ (\Phi \otimes 1\!\!1 + \eta \otimes \eta)$$

Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ *called* **contact metric structure**

The pair $(\mathcal{D}, J)$ *is a strictly pseudo-convex almost CR structure.*

Definition: *The structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *is* **K-contact** *if* $\pounds_\xi g = 0$ *(or* $\pounds_\xi \Phi = 0$*). It is* **Sasakian** *if in addition* $(\mathcal{D}, J)$ *is integrable.*

Note: Quasi-regular $\Rightarrow$ *K-contact, a torus of Reeb type* $\Rightarrow$ *K-contact. (Rukimbira)*

$\mathcal{D}$ *contact structure.* $\mathcal{J}(\mathcal{D})$ *space of strictly pseudoconvex CR structures on* $\mathcal{D}$

$\mathcal{J}(\mathcal{D})$ *is a contractible smooth Fréchet manifold*

Define group $\mathfrak{CR}(\mathcal{D}, J) \subset \mathfrak{Diff}(M)$

$\{\phi \mid \phi_* \mathcal{D} \subset \mathcal{D},\ \phi_* J = J \phi_*\}$

group of CR transformations

Thm: $\mathfrak{CR}(\mathcal{D}, J)$ *is a compact Lie group except for standard CR structure on* S^{2n+1} *when it is* $SU(n+1,1)$. *(Schoen, Lee, Webster, Frances)*

Note: $\mathfrak{r} =$ *rank of* $\mathfrak{CR}(\mathcal{D}, J)$

$\mathfrak{Con}(M, \mathcal{D})$ acts on $\mathcal{J}(\mathcal{D})$ with isotropy group $\mathfrak{CR}(\mathcal{D}, J)$ at $J \in \mathcal{J}(\mathcal{D})$.

Allows us to define

$\mathfrak{Q} : \mathcal{J}(\mathcal{D}) \to \mathfrak{SC}_T(\mathcal{D})$

by $\mathfrak{Q}(J) =$ the unique conjugacy class of maximal torus T in $\mathfrak{CR}(\mathcal{D}, J)$, hence a unique conjugacy class of maximal torus in $\mathfrak{Con}(M, \mathcal{D})$.

$\mathfrak{Q}$ is far from injective, but if $\mathcal{D}$ is K-contact type, it surjects onto classes of Reeb type.

Sasaki cones

$\mathfrak{t}_{\mathfrak{r}}^{+}(\mathcal{D}, J) = \{\xi \in \mathfrak{t}_{\mathfrak{r}} \mid \eta'(\xi) > 0, \}$
s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ *is Sasakian*

- *finite dim'l moduli of Sasakian structures within CR structure* $\kappa(\mathcal{D}, J) = \mathfrak{t}_{\mathfrak{r}}^{+}(\mathcal{D}, J)/\mathcal{W}(\mathcal{D}, J)$

A given $\mathcal{D}$ can have many Sasaki cones $\mathfrak{t}_{\mathfrak{r}}^{+}(\mathcal{D}, J_k)$ labelled by almost complex structures, and $\mathfrak{r} = \mathfrak{r}(k)$.

• **Extremal Sasakian metrics**

(B-,Galicki,Simanca)

$E(g) = \int_M s_g^2 d\mu_g,$

• Deform CR structure

Vary $\eta \mapsto \eta + it\partial\bar{\partial}\varphi,\ \varphi$ basic, gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic. $s_g =$ scalar curvature. Only applies to tori of Reeb type.

case: constant scalar curvature Sasakian (cscS) . If $c_1(\mathcal{D}) = 0$ $\Rightarrow$ Sasaki-η-Einstein (SηE)

$\mathrm{Ric}_g = ag + b\eta\otimes\eta,\ a, b$ constants.

Sasaki-Einstein (SE) $b = 0$

Extremal Set $\mathfrak{e}(\mathcal{D}, J)$

$\mathfrak{e}(\mathcal{D}, J) \subset \mathfrak{t}_{\mathfrak{r}}^{+}(\mathcal{D}, J)$ is open in Sasaki cone B-,Galicki,Simanca

If $\mathcal{S} = \mathcal{S}_1 \in \mathfrak{e}(\mathcal{D}, J)$ then entire ray $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a) \in \mathfrak{e}(\mathcal{D}, J)$

When is $\mathfrak{e}(\mathcal{D}, J) = \mathfrak{t}_{\mathfrak{r}}^{+}(\mathcal{D}, J)$?

Many ex's if $\dim \mathfrak{t}_{\mathfrak{r}}^{+}(\mathcal{D}, J) = 1$

- If $\dim\ \kappa(\mathcal{D}, J) > 1$, there are only two known cases when $\mathfrak{e}(\mathcal{D}, J) = \mathfrak{t}_{\mathfrak{r}}^{+}(\mathcal{D}, J) > 1$.

(1) standard CR structure on S^{2n+1}

Toric (dim $\kappa(\mathcal{D}, J) = n + 1$.)

$\kappa(\mathcal{D}, J) = \{\mathbf{w} = (w_0, \cdots, w_n) \in \mathbb{R}^{n+1} \mid w_0 \leq w_1 \leq \cdots \leq w_n\}$

All $\mathcal{S}_{\mathbf{w}}$ have extremal representatives, but only Φ-sect. curv. $c > -3$ has (cscS), and only the round sphere ($c = 1$) is SE.

(B,Galicki,Simanca)

(2) The Heisenberg group $\mathfrak{H}^{2n+1}$ with standard CR structure (non-compact), dim $\kappa(\mathcal{D}, J) = n$. (B-)

All $\mathcal{S} \in \kappa(\mathcal{D}, J)$ have extremal representatives, but there is only one with constant scalar curvature, SηE with Φ-holomorphic curvature $= -3$. Here transverse homothety is induced by diffeomorphism.

Probably $\mathfrak{e}(\mathcal{D}, J) = \mathfrak{t}_{\mathfrak{r}}^{+}(\mathcal{D}, J)$ also holds for standard CR structure on the hyperbolic ball $B_{\mathbb{C}}^{n} \times \mathbb{R}$. Here Φ-sect. curv. $c < -3$ is (cscS).

orbifold Boothby-Wang: Manifold M compact with quasi-regular contact form $\eta \Rightarrow$ quotient $\pi : M \to \mathcal{B} = M/\mathcal{F}_\xi$ symplectic orbifold $(\mathcal{B}, \omega)$ with $\pi^*\omega = d\eta$.

M K-contact $\Rightarrow \mathcal{B}$ almost Kähler.

Converse: $\mathcal{B} = M/\mathcal{F}_\xi$ almost Kähler orbifold. ω Kähler form with $[\omega] \in H^2_{orb}(\mathcal{B}, \mathbb{Z})$. Total space M of S^1 orbibundle over $\mathcal{B}$ has K-contact structure. $(\mathcal{B}, \omega)$ is projective algebraic orbifold $\iff$ (ξ, η, Φ, g) is Sasakian. (B-,Galicki)

$\mathcal{S} = (\xi, \eta, \Phi, g)$ quasi-regular K-contact structure. $\Xi \approx S^1$ the central one-parameter subgroup generated by Reeb vector field ξ.

Thm:(B-) There is an isomorphism of Fréchet Lie groups
$\mathfrak{Con}(M, \eta)_0/\Xi \approx \mathfrak{Ham}(\mathcal{B}, \omega)$.
$(\mathcal{B}, \omega)$ symplectic orbifold
Here $\mathfrak{Ham}(\mathcal{B}, \omega) =$ group of Hamiltonian isotopies, and $\pi^*\omega = d\eta$.
This generalizes result of Banyaga in the regular case.

Thm:(B-) Two tori $\hat{T}_1$ and $\hat{T}_2$ in $\mathfrak{Ham}(\mathcal{B},\omega)$ are conjugate $\Longleftrightarrow$ lifed tori $T_1 \times \Xi$ and $T_2 \times \Xi$ are conjugate in $\mathfrak{Con}(M,\mathcal{D})$.

This identifies conjugacy classes of maximal tori in $\mathfrak{Ham}(\mathcal{B},\omega)$ with conjugacy classes of maximal tori of Reeb type in $\mathfrak{Con}(M,\mathcal{D})$ with Reeb field ξ.

Compatible almost complex structure $\hat{J}$ on $(\mathcal{B},\omega) \Rightarrow J \in \mathcal{J}(\mathcal{D})$.

$\hat{J} \mapsto$ conjugacy class of maximal tori in $\mathfrak{Ham}(\mathcal{B},\omega) \Rightarrow \mathfrak{SC}_T(\mathcal{B},\omega)$

Get a commutative diagram

$$\begin{array}{ccc} \mathcal{J}(\mathcal{D};\mathfrak{r}+1) & \xrightarrow{\mathfrak{Q}_{\mathcal{D},\mathfrak{r}+1}} & \mathfrak{S}\mathfrak{C}_T(\mathcal{D};\mathfrak{r}+1) \\ \uparrow & & \uparrow \\ \mathcal{J}(\mathcal{B},\omega;\mathfrak{r}) & \xrightarrow{\mathfrak{Q}_{\mathcal{B},\omega;\mathfrak{r}}} & \mathfrak{S}\mathfrak{C}_T(\mathcal{B},\omega;\mathfrak{r}) \end{array}$$

When is $\mathfrak{Q}_{\mathcal{D},\mathfrak{r}+1}$ surjective?

It is in **Toric case**: $\mathfrak{r}=n$.

moment map $\mu:\mathcal{B}\to\mathfrak{t}_n^*$ whose image is a convex rat'l simple polytope Δ with integer on facets.

LT-polytopes (Lerman/Tolman).

Image of contact moment map $\mu:M\to\mathfrak{t}_{n+1}^*$ is a convex rational polyhedral cone.

LT Thm: 1-1 correspondence between isomorphism classes of toric symplectic orbifolds and isomorphism classes of LT-polytopes under $GL(n, \mathbb{Z})$ transformations.

Thm: 1-1 correspondence between isomorphism classes of toric contact manifolds of Reeb type and isomorphism classes of good rational polyhedral cones under $GL(n+1, \mathbb{Z})$ transformations.

Toric symplectic structures are all Kähler. Toric contact structures of Reeb type are all Sasakian. (B-,Galicki)

In the toric case the orbifold Boothby-Wang projection
$(M, \xi, \eta, \Phi) \to (\mathcal{B}, \omega, \hat{J})$
represented by cutting the polyhedral cone by the characteristic hyperplane given by $\eta(\xi) = 1$. Conversely, get the the polyhedral cone by constructing a cone over the LT-polytope.

Toric Examples

• $\mathcal{B} = S^2 \times S^2$, $\quad k_1 \geq k_2 > 0$.

$\omega = k_1\omega_1 + k_2\omega_2$

(Karshon) $\mathfrak{n}(S^2 \times S^2, \omega; 2) = \lceil \frac{k_1}{k_2} \rceil$

$\lceil r \rceil$ is smallest integer $\geq r$.

(Lerman) $\mathfrak{n}_R(\mathcal{D}_{k_1,k_2}; 3) \geq \lceil \frac{k_1}{k_2} \rceil$

(J. Pati) $\mathcal{D}_{k_1,k_2}$ are distinct contact structures on $S^2 \times S^3$ for distinct (k_1, k_2). $\gcd(k_1, k_2) = 1$

$c_1(\mathcal{D}_{k_1,k_2}) = 2(k_1 - k_2)[\gamma]$

(Wang/Ziller,B-/Galicki)

complex structures $\hat{J}$ Hirzebruch surfaces (even)

Take $k_1 = 5, k_2 = 1$. There are 5 conjugacy classes coming from transverse complex structures $J_k,\ k = 0, \ldots 4$. Arise from the Hirzebruch surfaces, S_{2k} giving rays of extremal Sasakian metrics g_k in $\mathcal{D}_{5,1}$ (from Calabi on S_{2n}), each belonging to a different Sasaki cone $\mathfrak{t}_3^+(k)$ lying in five 3-dim'l vector subspaces $\mathfrak{t}_3(k) \subset \mathfrak{con}(S^2 \times S^3, \mathcal{D}_{5,1})$ intersect in $\{\xi\}$. $\exists$ open set of rays of extremal metrics in $\mathfrak{t}_3^+(k)$. Of g_k only g_0 has cscS. Maybe cscS $\in \mathfrak{t}_3^+(k)$.

- $\mathcal{B} = \widetilde{\mathbb{CP}}^2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, $l > e > 0$.

$\omega_{l,e}$, E exceptional divisor, L divisor of $\mathbb{CP}^1$ such that symplectic area of $L(E)$ is $2\pi l(e)$, resp.

(Karshon) $\mathfrak{n}(\widetilde{\mathbb{CP}}^2, \omega; 2) = \lceil \frac{e}{l-e} \rceil$

gives $\mathfrak{n}_R(\mathcal{D}_{l,e}; 3) \geq \lceil \frac{e}{l-e} \rceil$

$c_1(\mathcal{D}_{l,e}) = (l-2e)[\gamma]$

complex structures $\hat{J}$ Hirzebruch surfaces (odd). Two cases:

l even $\Rightarrow$ manifold is $S^2 \times S^3$

l odd $\Rightarrow$ manifold is X_∞, the nontrivial S^3-bundle over S^2.

All have extremal Sasakian metrics by Calabi, but not cscS.

- In both of the above cases Moduli space of extremal Sasakian metrics is non-Hausdorff.
- Generally, there is $\mathbb{Z}^4$'s worth of toric contact structures on $S^2 \times S^3$ and X_∞, gcd conditions.
- Are there non-intersecting Sasaki cones in same $\mathcal{D}$?
- Toric Sasakian structures on $k(S^2 \times S^3)$ and $X_\infty \# k(S^2 \times S^3)$ (B-,Galicki,Ornea)

- Polygon Spaces

Pol(α) $=$ quotient by $SO(3)$ of configurations in $\mathbb{R}^3$ of a polygon with m edges of length $\alpha_1, \ldots \alpha_m$ Pol(α) smooth compact symplectic manifold of dim $2(m-3)$. Construct maximal Hamiltonian tori. Special case: 8-dim smooth symplectic manifold with 3 conjugacy classes of maximal tori all of different dimensions, $2, 3$, and 4, the latter being toric. (Hausmann-Tolman).Then consider S^1 bundles over Pol(α).

• **Toric** S^3.

Eliashberg: up to contactomorphism $\exists$ 2 contact structures

(1) contains round sphere Reeb (Sasaki) type.

$\mathfrak{n}(\mathcal{D},2) = \mathfrak{n}_R(\mathcal{D},2) = 1$

(2) overtwisted contact structure: non-Reeb type

$$\begin{aligned}\eta_k &= \cos(2k+\tfrac{1}{2})\pi t)d\theta_1 \\ &\quad + \sin(2k+\tfrac{1}{2})\pi t)d\theta_2\end{aligned}$$

Each k distinct toric contact structure. $\Rightarrow$ $\mathfrak{n}(\mathcal{D},2) = \aleph_0$ (Lerman)

$\mathfrak{n}_R(\mathcal{D}) = 0$

• **Toric, non-Reeb type**:

The unit sphere bundle in the cotangent bundle of a torus: $S(T^*T^{n+1}) \approx S^n \times T^{n+1}$ admits contact structure (Lutz)

all are toric with (Lerman)

$\mathfrak{n}(\mathcal{D}, n+1) = 1, \mathfrak{n}_R(\mathcal{D}) = 0$

$n = 2 \Rightarrow H^2(S^2, \mathbb{Z}^3) = \mathbb{Z}^3$ giving non-trivial T^3-bundles over S^2 these are all toric with

$\mathfrak{n}(\mathcal{D}, 3) = 1, \mathfrak{n}_R(\mathcal{D}) = 0$

(Lerman)

• **Obstructions** to **cscS metrics**

Sasaki-Futaki invariant

(B-,Galicki,Simanca)

Gauntlett,Martelli,Sparks,Yau use old estimate of Lichnerowicz to obtain new obstruction to SE metrics. Generalize to obstruction to cscS metrics. $\Rightarrow$ orbifold slope stability (Ross,Thomas)

Project: Correct notion of stability in Sasakian geometry.

General Reference:

C. P. Boyer and K. Galicki, Sasakian Geometry, OUP, 2008.

- Higher dimensions
- Many examples with 1 dimensional Torus in all odd dimensions admitting SE metrics. Here there is only one conjugacy class of maximal tori, and the moduli space of SE metrics is Hausdorff.