On Toric Contact Geometry

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• **Contact Manifold** M (compact). *A* **contact 1-form** η *such that*

$$\eta \wedge (d\eta)^n \neq 0.$$

defines a **contact structure**

$$\eta' \sim \eta \iff \eta' = f\eta$$

for some $f \neq 0$, *take* $f > 0$. *or equivalently a codimension 1 sub-bundle* $\mathcal{D} = \mathrm{Ker}\ \eta$ *of* TM *with a conformal symplectic structure.*

Unique vector field ξ, *called the* **Reeb vector field**, *satisfying*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

The **characteristic foliation** $\mathcal{F}_\xi$ *each leaf of* $\mathcal{F}_\xi$ *passes through any nbd* U *at most* k *times* $\Longleftrightarrow$ **quasi-regular**, $k = 1 \leftrightarrow$ *regular, otherwise* **irregular**

Quasi-regularity is strong, most contact 1-forms are irregular.

Contact bundle $\mathcal{D} \longrightarrow$ *choose* **almost complex structure** J *extend to* Φ *with* $\Phi\xi = 0$
with a compatible metric

$$g = d\eta \circ (\Phi \otimes 1\!\!1) + \eta \otimes \eta$$

Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ *called* **contact metric structure**

The pair $(\mathcal{D}, J)$ *is a strictly pseudo-convex almost CR structure.*

Definition: *The structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *is* **K-contact** *if* $\pounds_\xi g = 0$ *(or* $\pounds_\xi \Phi = 0$*). It is* **Sasakian** *if in addition* $(\mathcal{D}, J)$ *is integrable.*

Transverse Metric $g_{\mathcal{D}}$ *is Kähler*

Cone (Symplectization)

$C(M) = M \times \mathbb{R}^{+}$

symplectic form $d(r^2\eta),\ r \in \mathbb{R}^{+}$.

Cone Metric $g_C = dr^2 + r^2 g$

- g_C *is Kähler* $\iff$ g *is Sasaki* $\iff$ $g_{\mathcal{D}}$ *is Kähler.*

Sasaki-Kähler Sandwich

$\Leftrightarrow$

$\Leftrightarrow$

Toric Contact Manifold

$(M^{2n+1}, \mathcal{D})$, *effective action of torus* T^{n+1} *leaving* $\mathcal{D}$ *invariant.*

- *Completely Int. Ham. system*

Toric Symplectic Cone

$(M^{2n+1} \times \mathbb{R}^+, d(r^2\eta))$ *effective action of torus* T^{n+1} *leaving* $d(r^2\eta)$ *invariant, commuting with* $r\frac{\partial}{\partial r}$.

(1): **Reeb Type** *Reeb field* ξ *lies in* $\mathfrak{t}_{n+1}$, *Lie algebra of* T^{n+1}.

(2): $\xi \not\in \mathfrak{t}_{n+1}$. (less interesting)
Reeb type are Sasakian. B-/Galicki.

Other References: Banyaga/Molino, Lerman, Falcao de Moraes/Tomei.
Complete classification: Lerman.

Toric contact manifolds of Reeb type are classified by certain convex polyhedral cones in $\mathfrak{t}^*_{n+1}$ up to T^{n+1}-equivariant equivalence. (Lerman).

Simply connected 5-manifolds

Barden-Smale classification

$H_2(M^5,\mathbb{Z})$ *torsionfree*

$S^5, S^2\times S^3, X_\infty, k\#(S^2\times S^3),$

$X_\infty\#k\#(S^2\times S^3)$.

All admit toric contact structures of Reeb type. (B-/Galicki,Ornea)

All but S^5 *admit infinitely many.*

All can be obtained by **Symmetry Reduction**.

S^1 *reduction of* S^7 *(B-/Pati).*

$S^1_{\mathbf{p}}$*-action* $(z_1,z_2,z_3,z_4)\mapsto$

$(e^{ip_1\theta}z_1, e^{ip_2\theta}z_2, e^{-ip_3\theta}z_3, e^{-ip_4\theta}z_4)$
where $p_i \in \mathbb{Z}^+$, $\gcd(p_i, p_j) = 1$ *for* $i = 1, 2, j = 2, 3$.

moment map $\mu : S^7 \to \mathbb{R}$ *is* $\mu(\mathbf{z}) = p_1|z_1|^2 + p_2|z_2|^2 - p_3|z_3|^2 - p_4|z_4|^2$.

$\mu^{-1}(0)/S^1_{\mathbf{p}} = M^5$ *compact simply connected,* $H_2(M^5, \mathbb{Z}) = \mathbb{Z}$, *induced contact structure* $\mathcal{D}_{\mathbf{p}}$.

$M^5 = S^2 \times S^3$ *or* X_∞. *Which?*

$w_2(M^5) \equiv c_1(\mathcal{D}_{\mathbf{p}})$ mod 2. $\Rightarrow$

$M^5 = S^2 \times S^3 (X_\infty)$ *if* $p_1 + p_2 - p_3 - p_4$ *is even (odd).*

Special Case: $\mathcal{D}_{\mathbf{p}} = \mathcal{D}_{j,2k-j,l,l}$
$c_1(\mathcal{D}_{\mathbf{p}}) = 2(k-l)\gamma \Rightarrow S^2 \times S^3$.
Choose a Reeb vector field get S^1_ϕ*-action* $(z_1, z_2, z_3, z_4) \mapsto (e^{i(2k-j)\phi}z_1, e^{ij\phi}z_2, e^{il\phi}z_3, e^{il\phi}z_4)$
The quotient M^5/S^1_ϕ *is an orbifold Hirzebruch surface.*
Special case of our special case: $Y^{p,q} \approx S^2 \times S^3$. *Physicists: Gauntlett, Martelli, Sparks, Waldram.*
Infinitely many toric contact structures admit Sasaki-Einstein metrics. In our notation $\mathcal{D}_{p-q,p+q,p,p}$

with $\mathrm{gcd}(p,q)=1$ *and* $1\le q<p$.

$c_1(\mathcal{D}_{p-q,p+q,p,p})=0$. $\Leftarrow$ *SE*

$Y^{p,q}$ *and* $Y^{p',q'}$ *are toric contact equivalent* $\iff$ $(p',q')=(p,q)$.

Theorem: $Y^{p,q}$ *and* $Y^{p',q'}$ *are contact equivalent* $\iff$ $p'=p$.

Outline of Proof:

Equivalence when $p'=p$: *Amounts to proving that the 3-tori corresponding to* $Y^{p,q}$ *and* $Y^{p,q'}$ *belong to distinct conjugacy classes of maximal tori in the contactomorphism group* $\mathfrak{Con}(\mathcal{D}_{p-q,p+q,p,p})$. *Simplicity take* p *odd. The quotient* $Y^{p,q}/S^1_\phi$ *is orbifold Hirzebruch surface* (S_{2q},Δ_p)

$S_{2q} \approx S^2 \times S^2$, but with a twisted complex structure. Represent as $S_{2q} = \mathbb{P}(H^{2q} \oplus \mathbb{C})$, H is the hyperplane bundle over $\mathbb{P}^1$, $\mathbb{C}$ is the trivial line bundle.

$\Delta_p = (1 - \frac{1}{p})(E + F)$ is branch divisor with ramification index p, divisors $E \cdot E = 2q$ and $F \cdot F = -2q$.

$\mathbb{1} : (S_{2q}, \Delta_p) \to (S_{2q}, \emptyset)$ identity map is a Galois cover. $\emptyset$ means trivial orbifold structure. Karshon: symplectomorphism $S_{2q} \xrightarrow{K} S_{2q-2}$ with same symplectic form ω.

Get commutative diagram

$$\begin{array}{ccc} (S_{2q}, \Delta_p) & \stackrel{K_p}{\longrightarrow} & (S_{2q'}, \Delta_p) \\ \downarrow \mathbb{1} & & \uparrow \mathbb{1}^{-1} \\ (S_{2q}, \emptyset) & \stackrel{K}{\longrightarrow} & (S_{2q'}, \emptyset) \end{array}$$

K_p is an orbifold symplectomorphism. Neither K nor K_p are T^2-equivariant symplectomorphisms. Their toric structures belong to distinct conjugacy classes of maximal tori in $\mathfrak{Ham}(S_{2q}, \omega)$.

These toric symplectic structures on $S^2 \times S^2$ lift to toric contact structures

$\mathcal{D}_{p-q,p+q,p,p}$ and $\mathcal{D}_{p-q',p+q',p,p}$ on $S^2 \times S^3$ that are contactomorphic, but not T^3-equivariantly contactomorphic.

Non-equivalence if $p' \neq p$:

Contact homology

(Eliashberg,Givental,Hofer)

Morse theory on loop space $L(M)$.

$\mathcal{A}(\gamma) = \int_\gamma \eta$. Critical points are closed Reeb orbits.

Perturb $\eta \Rightarrow$ 4 isolated critical points = critical points of norm of T^2 moment map μ_2 on space of Reeb orbits. Do for each period T including multiplicity k.

$S_T =$ Reeb orbits of period T.

Chain complex C_* differential graded algebra generated closed Reeb orbits, coefficient $\in H_2(M, \mathbb{Z}) = \mathbb{Z}$.

contact homology $= H(C_*)$.

grading: determined by Conley-Zehnder (Robbin-Salamon) index

differential: moduli space of genus zero J-holomorphic curves in $C(M)$.

Theorem: If $p'_1 + p'_2 \not\equiv (p_1 + p_2) \ mod \ (p_1 + p_2 - p_3 - p_4)$ then $\mathcal{D}_{\mathbf{p}}$ and $\mathcal{D}_{\mathbf{p}'}$ are not contactomorphic.

Corollary: The contact structures $Y^{p,q}$ and $Y^{p',q'}$ are not contactomorphic if $p' \neq p$. ■

(B-,Pati) and contact homology also (Abreu,Macarini)

• **Extremal Sasakian metrics**
(B-,Galicki,Simanca)
$E(g) = \int_M s_g^2 d\mu_g$,
• Deform CR structure
Vary $\eta \mapsto \eta + td^c\varphi$, φ basic, gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic. $s_g =$ scalar curvature. Only applies to tori of Reeb type.
case: constant scalar curvature Sasakian (cscS) . If $c_1(\mathcal{D}) = 0$
$\Rightarrow$ Sasaki-η-Einstein (SηE)
$\mathrm{Ric}_g = ag + b\eta \otimes \eta$, a, b constants.
Sasaki-Einstein (SE) $b = 0$

Theorem: Every toric contact structure of Reeb type with $c_1(\mathcal{D}) = 0$ admits a unique Sasaki-Einstein metric (Futaki,Ono,Wang,Cho)

• In this case the obstructions to deforming the Monge-Ampère equation vanish.

General Reference: **C. P. B- and K. Galicki, Sasakian Geometry**, Oxford University Press, 2008.

Sasaki cones

$\mathfrak{t}_3^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_3 \mid \eta(\xi') > 0,\}$ s.t. $\mathcal{S} = (\xi', \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian

Contact structure $\mathcal{D}$ can have many Sasaki cones $\mathfrak{t}_3^+(\mathcal{D}, J_k)$ labelled by almost complex structures J_k. These form a bouquet $\bigcup_k \mathfrak{t}_3^+(\mathcal{D}, J_k)$ cones intersect in 2-dim subspace. For $Y^{p,q}$ each cone contains extremal rays and SE metric. Number of cones in bouquet is Euler phi function $\phi(p)$.

Questions:

1. For which toric varieties V are there completely integrable Hamiltonian systems on T^*V?

2. For which toric varieties V^{2n} is there a separation of variables with n ignorable coordinates for the Hamilton-Jacobi equation on T^*V? for the Laplace-Beltrami equation?