Obstructions for Extremal Sasaki Metrics

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4. I learned a lot from **Gabor's** book: **An Introduction to Extremal Kähler Metrics**.

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7. Most contact forms in a contact structure $\mathcal{D}$ are **irregular**
8. We can choose a **compatible almost complex structure** J on $\mathcal{D}$, that is one that satisfies the two conditions
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- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes 1\!\!1) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.

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The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi\Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M, \mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M) = M\times\mathbb{R}^+, dr^2 + r^2 g)$ is Kähler.
- Some properties of a Sasaki manifold $(M, \mathcal{S})$ of dimension $2n + 1$:
  1. Any Sasaki structure $\mathcal{S}$ has at least an S^1 symmetry.
  2. The **characteristic foliation** $\mathcal{F}_\xi$ is Riemannian, that is, a **Riemannian flow**.
  3. If $\mathcal{S}$ is irregular, then the closure $\bar{\mathcal{F}}_\xi$ is a **torus** T^k of dimension $1 \leq k \leq n + 1$.
  4. The metric g is **bundle-like** and the leaves of $\mathcal{F}_\xi$ (orbits of ξ) are **totally geodesic**.
  5. In the quasi-regular case $(M, \mathcal{S})$ is an S^1 orbibundle over a **projective algebraic variety** with an additional **orbifold structure**.
  6. The **Ricci curvature** of g satisfies $\mathrm{Ric}_g(X, \xi) = 2n\eta(X)$ for any vector field X.

The Sasaki Cone and the Affine cone

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- the T **invariant index character** as a Hilbert series for $\xi \in \mathfrak{t}^+$, viz.

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where χ is the transverse Futaki-Mabuchi extremal vector field and $\langle \cdot, \cdot \rangle$ is their inner product.

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- (van Coevering) **Extremal Sasaki metrics** are **unique** up to **transverse holomorphic autmorphisms**.

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If f is a **homogeneous holomorphic function** *with* **weight** *$\alpha \in \mathfrak{t}^*$ satisfying the GMSY Theorem and $\alpha|_{T\Sigma} = 0$, then the* **entire Sasaki cone** *is* **obstructed** *from admitting extremal Sasaki metrics.*

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- Let f be a **weighted homogeneous polynomial** in $\mathbb{C}^{n+1}$ of degree d with **weight vector** $\mathbf{w}=(w_0,\ldots,w_n)$, i.e. $f(\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n)=\lambda^d f(z_0,\ldots,z_n)$ where $\lambda\in\mathbb{C}^*$.

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- Let f be a **weighted homogeneous polynomial** in $\mathbb{C}^{n+1}$ of degree d with **weight vector** $\mathbf{w} = (w_0, \ldots, w_n)$, i.e. $f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n)$ where $\lambda \in \mathbb{C}^*$.
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Theorem (B-,van Coevering)

If the inequality $\sum_{i=0}^k w_i - w_0 n + \frac{d}{2}(n-k-2) \geq 0$ holds, then there are **no extremal Sasaki metrics** *in the* **entire Sasaki cone** *of the link L_f.*

- Our results give statements about the **moduli space** $\mathcal{M}^c_{+,0}$ of classes of positive Sasakian structures with $c_1(\mathcal{F}_\xi) = 0$.

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- If $\mathfrak{C}(\mathcal{S}) = n = \frac{\dim M-1}{2}$ then either $\mathfrak{e} = \mathfrak{t}^+$ or $\mathfrak{e} = \emptyset$. Here $\dim \mathfrak{t}^+ = 1$.

Table of Manifolds having Sasaki Cones with no Extremal Metrics

Diffeo-(homeo)-morphism Type	f	$\mathfrak{C}(\mathcal{S})$
$S^{2n}\times S^{2n+1}$	$z_0^{8l}+z_1^2+\cdots+z_{2n+1}^2,\ n,l\geq 1$	n
$S^{2n}\times S^{2n+1}\#\Sigma_1^{4n+1}$	$z_0^{8l+4}+z_1^2+\cdots+z_{2n+1}^2=0,\ n\geq 1, l\geq 0$	n
Unit tangent bundle of S^{2n+1}	$z_0^{4l+2}+z_1^2+\cdots+z_{2n+1}^2,\ n>1, l\geq 1$	n
Homotopy sphere Σ_k^{4n+1}	$z_0^{2k+1}+z_1^2+\cdots+z_{2n+1}^2,\ n>1, k\geq 1$	n
Homotopy sphere Σ_k^{4n-1}	$z_0^{6k-1}+z_1^3+z_2^2+\cdots+z_{2n}^2,\ n\geq 2, k\geq 1$	n
Rat. homology sphere $H_{2n}\approx\mathbb{Z}_3$	$z_0^k+z_1^3+\cdots+z_{2n}^2,\ n,k>1$	n
$2k(S^{2n+1}\times S^{2n+2}),\ \ D_{n+1}(k)$	$z_0^{2(2k+1)}+z_1^{2k+1}+z_2^2+\cdots+z_{2n+2}^2,\ n,k\geq 1$	$n+1$
$\#m(S^2\times S^3), m=\gcd(p,,q)-1$	$z_0^p+z_1^q+z_2^2+z_3^2,\ \ p\geq 2q$ or $q\geq 2p$	1

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$S^{2n} \times S^{2n+1} \# \Sigma_1^{4n+1}$	$z_0^{8l+4} + z_1^2 + \cdots + z_{2n+1}^2 = 0, \; n \geq 1, l \geq 0$	n
Unit tangent bundle of S^{2n+1}	$z_0^{4l+2} + z_1^2 + \cdots + z_{2n+1}^2, \; n > 1, l \geq 1$	n
Homotopy sphere Σ_k^{4n+1}	$z_0^{2k+1} + z_1^2 + \cdots + z_{2n+1}^2, \; n > 1, k \geq 1$	n
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Rat. homology sphere $H_{2n} \approx \mathbb{Z}_3$	$z_0^k + z_1^3 + \cdots + z_{2n}^2, \; n, k > 1$	n
$2k(S^{2n+1} \times S^{2n+2}), \; D_{n+1}(k)$	$z_0^{2(2k+1)} + z_1^{2k+1} + z_2^2 + \cdots + z_{2n+2}^2, \; n, k \geq 1$	$n+1$
$\# m(S^2 \times S^3), m = \gcd(p, , q) - 1$	$z_0^p + z_1^q + z_2^2 + z_3^2, \; p \geq 2q$ or $q \geq 2p$	1

- Σ_1^{4n+1} is the **Kervaire sphere** which is **exotic** when $4n + 2 \neq 2^i - 2$.

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$2k(S^{2n+1} \times S^{2n+2}), \; D_{n+1}(k)$	$z_0^{2(2k+1)} + z_1^{2k+1} + z_2^2 + \cdots + z_{2n+2}^2, \; n, k \geq 1$	$n+1$
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Unit tangent bundle of S^{2n+1}	$z_0^{4l+2}+z_1^2+\cdots+z_{2n+1}^2,\ n>1, l\geq 1$	n
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Thank You

THANK YOU FOR YOUR ATTENTION