EXTREMAL SASAKIAN GEOMETRY

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University of New Mexico

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Inaugural Geometry Lectures,
CIMAT, Merida, Mexico

OUTLINE OF LECTURES

1. **LECTURE 1:** Fundamentals of Sasakian Geometry

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   - Fundamental Problems

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   - Critical Points of E-H Functional
   - Convexity of E-H Functional

IMPORTANT ANALOGIES

:

$$\text{Even Dimension} \longleftrightarrow \text{Odd Dimension}$$
$$\text{Symplectic Geometry} \longleftrightarrow \text{Contact Geometry}$$
$$\text{Kähler Geometry} \longleftrightarrow \text{Sasakian Geometry}$$

Some Fundamental Problems

I begin by giving some problems to keep in mind during the lectures. We can give only partial answers to these problems and usually only for particular cases. The idea is to use these as a guiding principle.

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   - Determine those having distinct underlying **CR structures**.

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7. Most contact forms in a contact structure $\mathcal{D}$ are **irregular**

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for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So {oriented contact 1-forms in $\mathcal{D}$} $\approx C^\infty(M)^+$
4. The pair $(M, \mathcal{D})$ is called a **contact manifold**.
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$$\eta(\xi) = 1, \qquad \xi \rfloor d\eta = 0.$$
6. The **characteristic foliation** $\mathcal{F}_\xi$ is the 1-dim'l foliation defined by ξ: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k = 1$; otherwise, it is **irregular**. We also say that the **contact form** η is **quasi-regular, regular, irregular**.
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Contact Geometry

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- **Transverse homothety**: If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a Sasakian structure, so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for every $a \in \mathbb{R}^+$ with $g_a = ag + (a^2 - a)\eta \otimes \eta$. So Sasakian structures come in **rays**.

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- But generally Ψ only exists locally near the boundary $\partial W = M$.
- However, when we can smooth the singularity at the cone point, Ψ will be globally defined and we have a **Liouville filling** for which the Kähler form ω is exact. This is the case for **Brieskorn manifolds**.

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- Under various technical assumptions, $\chi_m(W)$ exists and is a **contact invariant** independent of the Liouville filling which allows us to distinguish components of the **Sasaki moduli** space.

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- A Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **positive (negative)** if $c_1(\mathcal{F}_\xi) > 0$ ($c_1(\mathcal{F}_\xi) < 0$) or **null Sasakian** if $c_1(\mathcal{F}_\xi) = 0$. Otherwise $\mathcal{S}$ is **indefinite**.

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- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE. More later.

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$$\mathbf{H}(\xi) = \frac{\mathbf{S}_\xi^{n+1}}{\mathbf{V}_\xi^n}$$

where $\mathbf{V}$ is the total volume.

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- Sasaki version of the Donaldson-Tian-Yau **conjecture**: **cscS** $\iff$ cone is **K-polystable**. More later.

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- The link L_f is endowed with a natural **quasi-regular Sasakian structure** inherited as a Sasakian submanifold of the sphere S^{2n+1} with its "weighted" Sasakian structure $\mathcal{S}_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$.

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- There is a fibration of $(S^{2n+1} - L_f) \longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a **bouquet of n-spheres** $S^n \vee \cdots \vee S^n$.

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- **Orlik [1978]** proposed an algorithmic way of **computing torsion** of any L_f. He made a **conjecture** that his algorithm always produces correct answer. The conjecture was later proved by **Randell [1980]** for special links such as **Brieskorn-Pham links**. In full generality the conjecture is still open. However, it is known to hold for dimension 3 and it follows from the recent work of **Kollár** that it holds in dimension 5 **B-,Galicki,Simanca**.

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- The quotient of S^{2n+1} by the "**weighted** S^1-action" generated by the vector field $\xi_{\mathbf{w}}$ is the weighted projective space $\mathbb{P}^n_{\mathbb{C}}(\mathbf{w}) = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*_{\mathbf{w}}$.

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- Given the differentials in the **spectral sequence** of the fibrations $M_i \longrightarrow \mathrm{B}\mathcal{Z}_i \longrightarrow \mathrm{B}S^1$, use the commutative diagram to compute the **cohomology ring** of the **contact** manifold $M_1 \star_{l_1,l_2,\mathbf{w}} M_2$. Examples later.

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- We will see later how to deform in the **Sasaki cone** to obtain **Sasaki-Einstein metrics** when only one of the M_i has an **SE metric**.

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- For every $\mathbb{Q}$-divisor D numerically equivalent to $-K^{orb}_{\mathcal{Z}}$ there exists a resolution of singularities $\mu : Y \longrightarrow \mathcal{Z}$ of D such that for some $\gamma \in (\frac{n}{n+1}, 1)$ the multiplier ideal sheaf $\mathcal{J}(\gamma D) = \mu_*\mathcal{O}_Y(K_Y - \mu^* K^{orb}_{\mathcal{Z}} - \lfloor \mu^*\gamma D \rfloor)$ is the full structure sheaf $\mathcal{O}_{\mathcal{Z}}$, where $\lfloor\cdot\rfloor$ denotes the **round-down** and $K^{orb}_X = K_X + \sum(1 - \frac{1}{m_i})D_i$ where $\Delta = \sum(1 - \frac{1}{m_i})D_i$ with a D_i a **branch divisor** with **ramification index** m_i. This condition is known as **Kawamata log terminal** or **klt** for short, and guarentees a **Kähler-Einstein orbifold metric** on $\mathcal{Z}$; hence, a **Sasaki-Einstein metric** on M.

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- **Obstructions**: Using an estimate of Lichnerowicz, Gauntlett,Martelli,Sparks,Yau proved: If $(\sum_{i=0}^{n} w_i - d) > n \min_i w_i$ then $L_{f_{\mathbf{w}}}$ **cannot** admit a **Sasaki-Einstein metric**.

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- For example there are **better estimates in dimension five** (B-,Galicki,Nakamaye, Kollár, B-,Nakamaye).
- For **Brieskorn-Pham polynomials** we have the estimate

$$1 < \sum_{i=0}^{n} \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_{i,j}\Big\{\frac{1}{a_i}, \frac{1}{b_i b_j}\Big\}.$$

 where $b_j = \gcd(a_j, C^j)$ with $C^j = \mathrm{lcm}(a_i : i \neq j)$ implying the existence of a **Sasaki-Einstein metric** (B-,Galicki,Kollár). Again this is far from a necessary condition.
- **Obstructions**: Using an estimate of Lichnerowicz, Gauntlett,Martelli,Sparks,Yau proved: If $(\sum_{i=0}^{n} w_i - d) > n \min_i w_i$ then $L_{f_{\mathbf{w}}}$ **cannot** admit a **Sasaki-Einstein metric**.
- Ghigi-Kollár: For a **Brieskorn-Pham polynomial** $L(\mathbf{a})$ with the components $a_0, \cdots, a_n$ of **a pairwise relatively prime**, $L(\mathbf{a})$ admits a **Sasaki-Einstein metric** if and only if

$$1 < \sum_{i=0}^{n} \frac{1}{a_i} < 1 + n \min_i \Big\{\frac{1}{a_i}\Big\}.$$

Sasaki-Einstein Metrics in Dimension Five

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- There are also many **Sasaki-Einstein metrics** on **mixed Smale manifolds** of the form $kM_\infty \# nM_m$ (B-,Galicki; Kollár; B-,Nakamaye) of course satisfying the torsion constraints.

Sasaki-Einstein Metrics on Homotopy Spheres

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- Example: the sequence $\mathbf{a} = (2, 3, 7, 43, 43 \cdot 39)$ gives a $2(44 + 4 - 5) = 86$ real parameter family of **Sasaki-Einstein metrics** on the **exotic sphere** $\Sigma_6 \in bP_8$ i.e. 6 times the Milnor generator.

Sasaki-Einstein Metrics in Abundance: the Sylvester Sequence

- The above sequence arises from a *boundary sequence* called **Sylvester's sequence** determined by the recursion relation $c_{k+1} = 1 + c_1 \cdots c_k = c_k^2 - c_k + 1$ beginning with $c_1 = 2$. It starts as $2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots$. The importance of this sequence is that it satisfies $\sum_{i=0}^{n} \frac{1}{c_i} = 1 - \frac{1}{c_0 \cdots c_n}$. This sequence does not satisfy the **klt conditions**. So we choose numbers that are less than the last number of the sequence, so it will satisfy a **klt** condition and also certain gcd conditions. There are many such sequences, so

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- However, a **Sasaki-Einstein metric** is unique in the space $\mathfrak{F}(\xi)$ up to a transverse holomorphic transformation (Nitta,Sekiya).

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LECTURE 3: Join Construction and Constant Scalar Curvature

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Recall the join construction

- We take $M_1 = M$ to be a regular Sasaki manifold and $M_2 = S^3_{\mathbf{w}}$, the three dimensional weighted sphere. That is the weighted Sasakian structure $\mathcal{S}_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ mentioned earlier with $\mathbf{w} = (w_1, w_2)$.

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- We take N to be a Kähler manifold with a constant scalar curvature metric. It is well known that $\mathbb{CP}^1[\mathbf{w}]$ has an extremal Kähler metric. It follows that $N \times \mathbb{CP}^1[\mathbf{w}]$ has an extremal Kähler metric; hence, $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ has an **extremal Sasaki metric**.

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- The join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ can also be realized as a **lens space bundle** over N with fiber the lens space $L(l_2; l_1 w_1, l_1 w_2)$.

Deforming the Reeb vector field in $\mathfrak{t}^+_{\mathbf{w}}$

- We deform the Reeb vector field in $\mathfrak{t}^+_{\mathbf{w}}$ to a new Reeb field $\xi_{\mathbf{v}}$ to look for **cscS** and other **extremal Sasaki metrics**. Here $\mathbf{v} = (v_1, v_2)$ is real valued vector.

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- The join gives rise to the following commutative diagram

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\begin{array}{ccccc}
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 & & \Big\downarrow \pi_L & & \\
 & & M_{l_1, l_2, \mathbf{w}} & & \\
 & \swarrow \pi_{\mathbf{w}} & & \searrow \pi_{\mathbf{v}} & \\
N \times \mathbb{CP}^1[\mathbf{w}] & & & & (S_n, \Delta) \\
 & p_{\mathbf{w}} \searrow & & \swarrow p_{\mathbf{v}} & \\
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- (S_n, Δ) can be realized as the projective bundle $\mathbb{P}(\mathbb{1} \oplus L_n)$ over N where L_n is a non-trivial complex line bundle where $n = l_1 \left(\frac{w_1 v_2 - w_2 v_1}{s}\right)$ with $s = \gcd(|w_1 v_2 - w_2 v_1|, l_2)$.

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- The fiber of (S_n, Δ) is $(\mathbb{CP}^1[\mathbf{v}]/\mathbb{Z}_m)$ with the branch divisor

$$\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2,$$

  where $m_i = m v_i$ with $m \in \mathbb{Z}^+$.

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Theorem (B-,Tønnesen-Friedman)

*Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-**join** with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in a 2-dimensional sub cone, the $\mathbf{w}$-cone, of the Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has **constant scalar curvature**.*

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- Relation to **CR Yamabe Problem** (Jerison and Lee): For a Sasaki structure the Webster pseudo-Hermitian metric coincides with the transverse Kähler metric. So a **CSC** Sasaki metric provides a solution to the CR Yamabe Problem. It is know that when the **CR Yamabe invariant** $\lambda(M)$ is **nonpositive**, the CSC metric is unique. However, when $\lambda(M) > 0$ there can be several CSC solutions. Our results provides many such examples.

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$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.

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- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$

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In each odd dimension $2p+3>5$ there exist countably infinite simply connected **toric contact** *manifolds $M_{l_1,l_2,\mathbf{w}}$ of Reeb type depending on 4 positive integers l_1,l_2,w_1,w_2 satisfying $\gcd(l_2,l_1w_i)=\gcd(w_1,w_2)=1$, and with* **integral cohomology ring**

$$H^*(M_{l_1,l_2,\mathbf{w}},\mathbb{Z})\approx\mathbb{Z}[x,y]/(w_1w_2l_1^2x^2,x^{p+1},x^2y,y^2)$$

where x,y are classes of degree 2 and $2p+1$, respectively. Furthermore, with l_1,w_1,w_2 fixed there are a **finite number of diffeomorphism types** *with the given cohomology ring. Hence, in each such dimension there exist simply connected smooth manifolds with* **countably infinite toric contact** *structures of Reeb type that are* **inequivalent** *as contact structures.*

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- In special cases we can determine the **diffeomorphism (homeomorphism, homotopy)** types.

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- From join construction get commutative diagram of fibrations:

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- The finiteness of the **diffeomorphism types** follows by Sullivan's rational homotopy theory. $M_{l_1,l_2,\mathbf{w}}$ is **formal**.

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- The distinct Sasaki cones in the bouquet $\mathfrak{B}_k$ correspond to distinct conjugacy classes of maximal tori in $\mathfrak{Con}(\mathcal{D}_{l_1,l_2,\mathbf{w}})$. Uses the work of Buşe on **equivariant Gromov-Witten invariants**.

S^3-bundles over Riemann surfaces [B-,Tønnesen-Friedman]

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
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- On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a ray of **CSC** Sasakian structures.
- When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(\mathcal{D}_k, J)$ on S^3-bundles over Σ_g are exhausted by **extremal Sasaki metrics**
- For $g \geq 20$ there are rays in $\kappa(\mathcal{D}_k, J)$ which admit **no** extremal Sasaki metrics.
- For any genus $g \geq 1$ and for each positive integer k, the contact manifold $(\Sigma_g \times S^3, \mathcal{D}_k)$ has a k-**bouquet** $\mathfrak{B}_k$ of 2-dimensional Sasaki cones.
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- The construction can be 'twisted' by reducible representations of the fundamental group $\pi_1(\Sigma_g)$. The irreducible representations of $\pi_1(\Sigma_g)$ give 1-dimensional Sasaki cones. They arise from stable rank two vector bundles and have CSC Sasaki metrics.

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- Some of the same type of results have been obtained on 5-manifolds whose fundamental group is a non-Abelian extension of $\pi_1(\Sigma_g)$ in Castañeda's thesis.

Example: Bouquet on $S^3 \times \Sigma_g$

- The contact structure $\mathcal{D}_{l_1,1,\mathbf{w}}$ on $S^3 \times \Sigma_g$ has a **4-bouquet** for all genera g.

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m	l_1	**w**
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- In all 4 cases we have $c_1(\mathcal{D}_{l_1,1,\mathbf{w}}) = (2-2g-l_1(w_1+w_2))\gamma = -2(g+3)\gamma$ where γ is a generator of $H^2(S^3 \times \Sigma_g, \mathbb{Z})$.

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- If, however, $g > 0$ with $l_2 > 1$ the topology of the join is not completely known, so we can say nothing about bouquets.

Sasakian K-Stability

- The notions of **K-stability, K-polystability, K-semistability** have been amply developed in Kählerian geometry. Here we only deal with **K-semistability**. A Sasaki manifold $(M, \mathcal{S})$ is said to be **K-semistable** if the affine cone $(C(M), I)$ is **K-semistable** (Collins,Székelyhidi).

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- The **Donaldson-Futaki invariant** of such a test configuration is defined by

$$Fut(Y_0, \xi, a) := \frac{\mathbf{V}_\xi}{n} D_a\left(\frac{\mathbf{S}_\xi}{\mathbf{V}_\xi}\right) + \frac{\mathbf{S}_\xi D_a \mathbf{V}_\xi}{n(n+1)\mathbf{V}_\xi}$$

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- We say that (Y,ξ) is **K-semistable** if for each $T_{\mathbb{C}}$ with $\xi\in\mathfrak{t}$ and any **$T_{\mathbb{C}}$-equivariant test configuration** $Fut(Y_0,\xi,a)\geq 0$. It is **K-polystable** if equality holds only if it is a product configuration, that is, Y_0 is isomorphic to Y.

The Einstein-Hilbert Functional and K-stability [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (1)

The set of **critical points** *of the Einstein–Hilbert functional* $\mathbf{H}_\xi$ *is the union of the* **zeros** *of the* **Sasaki-Futaki invariant** *and of the* **total transversal scalar curvature**. *In particular, if a Reeb vector field admits a* **compatible cscS metric** *then it is a* **critical point of the Einstein–Hilbert functional**.

- Proof: This follows immediately from

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\mathfrak{S}\mathfrak{F}_\xi \circ \Phi. \tag{1}$$

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If (Y,ξ) *is K-semistable then the Sasaki–Futaki invariant* $\mathbf{F}_\xi$ *vanishes identically, and* ξ *is a critical point of the Einstein-Hilbert functional* $\mathbf{H}(\xi)$. *Alternatively, if* ξ *is not a critical point of* $\mathbf{H}(\xi)$ *then* (Y,ξ) *is not K-semistable.*

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Stability Theorems continued [B-,Huang,Legendre,Tønnesen-Friedman]

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Suppose the **Sasaki cone** *is* **exhausted by extremal Sasaki metrics** *and that the total transverse scalar curvature does not vanish. Then the set of critical points of the* **Einstein-Hilbert functional** *is precisely the set of rays in the Sasaki cone with* **constant scalar curvature**. *In particular, in this case a Sasakian structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *has* **constant scalar curvature** *if and only if* (Y, ξ) *is* **K-semistable**.

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Theorem (4)

Let M be a regular Sasaki manifold with constant transverse scalar curvature, and consider the $S^3_{\mathbf{w}}$-join $M_{l_1,l_2,\mathbf{w}}$. Its $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$ has a cscS ray $\mathfrak{r}_\xi$ if and only if the Sasaki-Futaki invariant $\mathbf{F}_\xi$ vanishes on the Lie algebra $\mathfrak{t}_{\mathbf{w}} \otimes \mathbb{C}$. Then for $\xi \in \mathfrak{t}^+_{\mathbf{w}}$ the polarized affine cone (Y, ξ) associated to $M_{l_1,l_2,\mathbf{w}}$ is K-semistable if and only if the Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M_{l_1,l_2,\mathbf{w}}$ has constant scalar curvature (up to isotopy).

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- Proof: This follows from Theorem 2, work of Collins, Székelyhidi, and the vanishing of the Sasaki-Futaki invariant for **cscS metrics**. □

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Let M be a regular Sasaki manifold with constant transverse scalar curvature, and consider the $S^3_{\mathbf{w}}$-join $M_{l_1,l_2,\mathbf{w}}$. Its $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$ has a cscS ray $\mathfrak{r}_\xi$ if and only if the Sasaki-Futaki invariant $\mathbf{F}_\xi$ vanishes on the Lie algebra $\mathfrak{t}_{\mathbf{w}} \otimes \mathbb{C}$. Then for $\xi \in \mathfrak{t}^+_{\mathbf{w}}$ the polarized affine cone (Y, ξ) associated to $M_{l_1,l_2,\mathbf{w}}$ is K-semistable if and only if the Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M_{l_1,l_2,\mathbf{w}}$ has constant scalar curvature (up to isotopy).

- Outline of Proof: This involves a long calculation of the admissible Kähler construction together with the $S^3_{\mathbf{w}}$-Sasaki join construction which expresses the **Sasaki-Futaki invariant** in terms of certain **polynomial** on the Sasaki cone. □

Critical Points of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (5)

Assume that the **total transversal scalar curvature** *is sign definite and bounded away from* 0 *on each transversal set of the Sasaki cone, then there exists at least one* **Reeb vector field** *for which the* **Sasaki–Futaki invariant** $\mathfrak{S}\mathfrak{F}$ *vanishes identically.*

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- But the left hand side is invariant under scaling of ξ, so there exists an m_0 independent of ξ such that on a transverse subset Σ we have $\mathbf{H}(\xi) \geq m_\xi^{n+1}\mathbf{V}_\xi \geq m_o^{m+1}\mathbf{V}_\xi \quad \forall \xi \in \Sigma$.

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- So $\mathbf{H}(\xi)$ must reach a minimum somewhere in Σ and the variational formula implies $\mathfrak{S}\mathfrak{F} = 0$. □

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

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  2. If the transverse scalar curvature is positive and the first non-zero eigenvalue λ_1 of the Laplacian, restricted to the space of T–invariant functions is bounded below by $\frac{s_g^T}{2n+1}$, then **H** is convex near ξ.

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- The lemma follows from the second variational formula at a cscS metric:

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and the well known inequality $\|d(\eta(a))\|_g^2 \geq \lambda_1 \|\eta(a)\|_g^2$. □

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- This lemma leads to the general question: Are the rays of **constant scalar curvature Sasaki metrics** isolated in the Sasaki cone?

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THANK YOU! MUCHAS GRACIAS!