### **EXTREMAL SASAKIAN GEOMETRY**

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LECTURE 1: Fundamentals of Sasakian Geometry

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  - Fundamental Problems

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- Convexity of E-H Functional

# LECTURE 1: Fundamentals of Sasakian Geometry

#### **IMPORTANT ANALOGIES**

Even Dimension  $\longleftrightarrow$  Odd Dimension Symplectic Geometry  $\longleftrightarrow$  Contact Geometry Kähler Geometry  $\longleftrightarrow$  Sasakian Geometry

I begin by giving some problems to keep in mind during the lectures. We can give only partial answers to these problems and usually only for particular cases. The idea is to use these as a guiding principle.

#### **PROBLEMS**

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  - Determine the (pre)-moduli space of Sasakian structures with the same underlying CR structure.

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  - Determine those with Sasaki-Einstein metrics (SE). How many?
  - Determine the (pre)-moduli space of Sasakian structures with the same underlying CR structure.
  - Determine those having distinct underlying CR structures.

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$$\eta' \sim \eta \iff \eta' = f\eta$$

for some  $f \neq 0$ , take f > 0, or equivalently a codimension 1 subbundle  $\mathfrak{D} = \operatorname{Ker} \eta$  of TM with a conformal symplectic structure. So {oriented contact 1-forms in  $\mathfrak{D}$ }  $\approx C^{\infty}(M)^+$ 

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- The pair  $(M, \mathcal{D})$  is called a **contact manifold**.
- **3** If we choose a contact 1-form  $\eta$ , there is a unique vector field  $\xi$ , called the **Reeb vector field**, satisfying

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**3** The characteristic foliation  $\mathcal{F}_{\xi}$  is the 1-dim'l foliation defined by  $\xi$ : It is called quasi-regular if each leaf of  $\mathcal{F}_{\xi}$  passes through any nbd U at most k times. It is regular if k=1; otherwise, it is irregular. We also say that the contact form  $\eta$  is quasi-regular, regular, irregular.

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- **1** The characteristic foliation  $\mathcal{F}_{\xi}$  is the 1-dim'l foliation defined by  $\xi$ : It is called quasi-regular if each leaf of  $\mathcal{F}_{\xi}$  passes through any nbd U at most k times. It is regular if k=1; otherwise, it is irregular. We also say that the contact form  $\eta$  is quasi-regular, regular, irregular.
- Most contact forms in a contact structure D are irregular
- We can choose a compatible almost complex structure J on D, that is one that satisfies the two conditions

$$d\eta(JX, JY) = d\eta(X, Y)$$
  $d\eta(JX, Y) > 0$ 

for any sections X, Y of  $\mathfrak{D}$ .

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- **3** If we choose a contact 1-form  $\eta$ , there is a unique vector field  $\xi$ , called the **Reeb vector field**, satisfying

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- **1** The characteristic foliation  $\mathcal{F}_{\xi}$  is the 1-dim'l foliation defined by  $\xi$ : It is called quasi-regular if each leaf of  $\mathcal{F}_{\xi}$  passes through any nbd U at most k times. It is regular if k=1; otherwise, it is irregular. We also say that the contact form  $\eta$  is quasi-regular, regular, irregular.
- Most contact forms in a contact structure D are irregular
- We can choose a compatible almost complex structure J on D, that is one that satisfies the two conditions

$$d\eta(JX, JY) = d\eta(X, Y)$$
  $d\eta(JX, Y) > 0$ 

for any sections X, Y of  $\mathfrak{D}$ .

**1** The almost complex structure J extends to an endomorphism  $\Phi$  of TM satisfying  $\Phi \xi = 0$ .

• There is a 'canonical' compatible metric  $g = d\eta \circ (\Phi \otimes \mathbb{I}) + \eta \otimes \eta$ . Quadruple  $\mathcal{S} = (\xi, \eta, \Phi, g)$  called **contact metric structure**.

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The contact metric structure  $\mathcal{S}=(\xi,\eta,\Phi,g)$  is **K-contact** if  $\mathcal{L}_{\xi}g=0$  (or  $\mathcal{L}_{\xi}\Phi=0$ ). It is **Sasakian** if in addition  $(\mathcal{D},J)$  is integrable and the **transverse metric**  $g_{\mathcal{D}}$  is Kähler (**transverse holonomy** U(n)). In the latter case we say that the contact structure  $\mathcal{D}$  is of **Sasaki type**.

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  - **6** The Ricci curvature of g satisfies  $Ric_g(X, \xi) = 2n\eta(X)$  for any vector field X.

## Cone (Symplectization)

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- The (almost) complex structure I on C(M) is defined by  $I|_{\mathcal{D}} = J$ ,  $I\xi = \Psi$ ,  $I\Psi = -\xi$ .
- ullet  $g_{\mathcal{C}}$  is Kähler  $\Longleftrightarrow g$  is Sasaki  $\Longleftrightarrow g_{\mathcal{D}}$  is Kähler.
- Nested structures: Sasakian ⊂ sψCR ⊂ Contact
- Sasakian structure gives pseudo convex CR structure  $(\mathfrak{D}, J)$  and a transverse holomorphic structure  $(\xi, \bar{J})$ . The former fixes the contact structure while the latter fixes the characteristic foliation.
- Transverse homothety: If  $S=(\xi,\eta,\Phi,g)$  is a Sasakian structure, so is  $S_a=(a^{-1}\xi,a\eta,\Phi,g_a)$  for every  $a\in\mathbb{R}^+$  with  $g_a=ag+(a^2-a)\eta\otimes\eta$ . So Sasakian structures come in rays.
- The subset  $W^o = M \times (0,1] \subset C(M)$  is an open symplectic filling of  $(M, \mathcal{D})$ .
- When (M, D) is of Sasaki type one can perform a resolution of singularity at the cone point to obtain an honest filling W which is Kähler (holomorphic).
- But generally  $\Psi$  only exists locally near the boundary  $\partial W = M$ .
- However, when we can smooth the singularity at the cone point,  $\Psi$  will be globally defined and we have a **Liouville filling** for which the Kähler form  $\omega$  is exact. This is the case for **Brieskorn manifolds**.

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• Under various technical assumptions,  $\chi_m(W)$  exists and is a **contact invariant** independent of the Liouville filling which allows us to distinguish components of the **Sasaki moduli** space.

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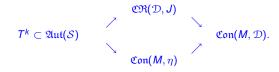
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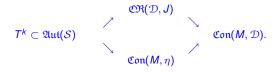


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$$\bigcup_{\alpha} \kappa(\mathfrak{D}, J_{\alpha})$$

of Sasaki cones,  $J_{\alpha} \in \mathcal{J}(\mathcal{D})$ ,  $\alpha$  ranges over distinct conjugacy classes.

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  - **3** Sasaki cone (reduced):  $\kappa(\mathcal{D}, J) = \mathfrak{t}^+_{\kappa}(\mathcal{D}, J)/\mathcal{W}$  where  $\mathcal{W}$  is the Weyl group of  $\mathfrak{CR}(\mathcal{D}, J)$
  - $\bullet$   $\kappa(\mathcal{D}, J)$  is finite dim'l moduli of Sasakian structures with underlying CR structure  $(\mathcal{D}, J)$ .
  - **1**  $\leq \dim \kappa(\mathcal{D}, J) \leq n+1$  and if  $\dim \kappa(\mathcal{D}, J) = n+1$ , M is toric Sasakian.
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  - a contact structure D of Sasaki type with a space of compatible CR (complex) structures J(D).
  - ② The space  $\mathcal{J}(\mathcal{D})$  is **contractible** and  $c_1(\mathcal{D})$  is independent of the choice of  $J \in \mathcal{J}(\mathcal{D})$ .
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- A Sasakian structure  $S = (\xi, \eta, \Phi, g)$  is **positive** (negative) if  $c_1(\mathcal{F}_{\xi}) > 0$  ( $c_1(\mathcal{F}_{\xi}) < 0$ ) or null Sasakian if  $c_1(\mathcal{F}_{\xi}) = 0$ . Otherwise S is indefinite.

• Let  $L_{\xi}$  be the **real line bundle** generated by  $\xi$  and consider the **short exact sequence** of vector bundles  $0 \longrightarrow L_{\xi} \longrightarrow TM \longrightarrow \nu(\mathcal{F}_{\xi}) \longrightarrow 0$ . A choice of **Sasakian structure**  $\mathcal{S} \in \mathfrak{F}(\xi)$  splits this sequence as  $TM = L_{\xi} \oplus \mathcal{D}$  with  $\mathcal{D} = \ker \eta$ .

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• The group  $\mathfrak{H}^T(\xi, \bar{J})$  is infinite dimensional, but the quotient  $\mathfrak{H}_0^T(\xi, \bar{J}) = \mathfrak{H}^T(\xi, \bar{J})/\Gamma(L_\xi)$  is a finite dimensional Lie group.

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$$\mathfrak{H}^{T}(\xi,\bar{J}) = \{ \phi \in \mathfrak{Fol}(M,\mathfrak{F}_{\xi}) \mid \bar{\phi}_{*} \circ \bar{J} = \bar{J} \circ \bar{\phi}_{*} \}.$$

- The group  $\mathfrak{H}^T(\xi, \overline{J})$  is infinite dimensional, but the quotient  $\mathfrak{H}_0^T(\xi, \overline{J}) = \mathfrak{H}^T(\xi, \overline{J})/\Gamma(L_{\xi})$  is a finite dimensional Lie group.
- The group  $\mathfrak{H}_0^T(\xi,\bar{J})$  does not preserve the contact structure  $\mathfrak{D}$ , but it does preserve its isotopy class.

- Let  $L_{\xi}$  be the real line bundle generated by  $\xi$  and consider the short exact sequence of vector bundles  $0 \longrightarrow L_{\xi} \longrightarrow TM \longrightarrow \nu(\mathcal{F}_{\xi}) \longrightarrow 0$ . A choice of Sasakian structure  $\mathcal{S} \in \mathfrak{F}(\xi)$  splits this sequence as  $TM = L_{\xi} \oplus \mathcal{D}$  with  $\mathcal{D} = \ker \eta$ .
- The transverse geometry is described on the quotient bundle  $\nu(\mathcal{F}_{\xi})$ . The endomorphism  $\Phi$  induces a complex structure  $\bar{J}$  on  $\nu(\mathcal{F}_{\xi})$ .
- If  $S \in \mathfrak{F}(\xi)$  the volume  $\mathbf{V}_S = \mathbf{V}_\xi = \int_M \eta \wedge (d\eta)^n$  and the total scalar curvature  $\mathbf{S}_S = \mathbf{S}_\xi = \int_M s_g \ \eta \wedge (d\eta)^n$  depend only on the class  $[d\eta]_B$ .
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- We denote by  $\mathfrak{h}_0^T(\xi, \bar{J})$  the Lie algebra of  $\mathfrak{H}_0^T(\xi, \bar{J})$ .

## Extremal Sasakian metrics (B-Galicki-Simanca)

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- The general extremal problem involves a 4th order PDE, whereas, the easier SE problem involves the Monge-Ampère equation, a second order PDE. More later.

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- Sasaki version of the Donaldson-Tian-Yau conjecture: cscS ← cone is K-polystable.
   More later.

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• The link  $L_f$  is endowed with a natural **quasi-regular Sasakian structure** inherited as a Sasakian submanifold of the sphere  $S^{2n+1}$  with its "weighted" Sasakian structure  $S_{\mathbf{W}} = (\xi_{\mathbf{W}}, \eta_{\mathbf{W}}, \Phi_{\mathbf{W}}, g_{\mathbf{W}})$ .

• There is a fibration of  $(S^{2n+1} - L_f) \longrightarrow S^1$  whose fiber F is an open manifold that is homotopy equivalent to a **bouquet of n-spheres**  $S^n \vee \cdots \vee S^n$ .

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  proved by Randell [1980] for special links such as Brieskorn-Pham links. In full generality
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  the recent work of Kollár that it holds in dimension 5 B-,Galicki,Simanca.

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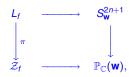
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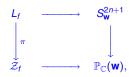
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Given the differentials in the spectral sequence of the fibrations M<sub>i</sub> → BZ<sub>i</sub> → BS<sup>1</sup>, use the commutative diagram to compute the cohomology ring of the contact manifold M<sub>1</sub> ⋆<sub>I<sub>1</sub>,I<sub>2</sub>,w</sub> M<sub>2</sub>. Examples later.

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For each i=1,2 let  $M_i^{2n_i+1}$  be a quasi-regular **SE manifold** whose base  $\mathcal{Z}_i$  has Fano index  $I_{\mathcal{Z}_i}$  and let  $I_i$  be their relative Fano indices. Assume also that  $\gcd(v_1 I_2, v_2 I_1) = 1$ . Then the **join**  $M_1 \star_{I_1,I_2} M_2$  is an **SE manifold** of dimension  $2(n_1 + n_2) + 1$ .

# The Sasaki-Einstein Semi-group

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 We will see later how to deform in the Sasaki cone to obtain Sasaki-Einstein metrics when only one of the M<sub>i</sub> has an SE metric.

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$$\frac{\det(\omega_{i\bar{j}}^T + i\partial_i\bar{\partial}_{\bar{j}}\phi)}{\det(\omega_{i\bar{j}}^T)} = e^{-t\phi + F}, \quad \omega_{i\bar{j}}^T + \partial_i\bar{\partial}_{\bar{j}}\phi > 0$$

for  $t \in [0, 1]$  by the continuity method Aubin, Yau, Siu, Tian, Nadel for Kähler manifolds, Demailly-Kollár, for **Kähler orbifolds**, El Kacimi-Alaoui for transverse to foliations.

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- For every  $\mathbb{Q}$ -divisor D numerically equivalent to  $-K_Z^{orb}$  there exists a resolution of singularities  $\mu: Y \longrightarrow \mathcal{Z}$  of D such that for some  $\gamma \in (\frac{n}{n+1}, 1)$  the multiplier ideal sheaf  $\mathcal{Z}(\gamma D) = \mu_* \mathcal{O}_Y(K_Y \mu^* K_Z^{orb} \lfloor \mu^* \gamma D \rfloor)$  is the full structure sheaf  $\mathcal{O}_Z$ , where  $\lfloor \cdot \rfloor$  denotes the round-down and  $K_X^{orb} = K_X + \sum (1 \frac{1}{m_i})D_i$  where  $\Delta = \sum (1 \frac{1}{m_i})D_i$  with a  $D_i$  a branch divisor with ramification index  $m_i$ . This condition is known as Kawamata log terminal or klt for short, and guarentees a Kähler-Einstein orbifold metric on Z; hence, a Sasaki-Einstein metric on M.

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- Ghigi-Kollár: For a Brieskorn-Pham polynomial  $L(\mathbf{a})$  with the components  $a_0, \cdots, a_n$  of a pairwise relatively prime,  $L(\mathbf{a})$  admits a Sasaki-Einstein metric if and only if

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• If a simply connected 5-manifold admits a **Sasaki-Einstein metric**, it must be a **Smale** manifold  $M^5$  that is either one of the following or connected sums of the following:  $M_{\infty} = S^2 \times S^3$ ,  $M_1 = S^5$ ,  $M_j$  which satisfies  $H_2(M_j, \mathbb{Z}) = \mathbb{Z}_j \oplus \mathbb{Z}_j$ ,  $M^5 = M_{k_1} \# \cdots \# M_{k_s}$ ,  $k_i$  divides  $k_{i+1}$  or  $k_{i+1} = \infty$ .

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- In dimension five there are Sasaki-Einstein metrics on the connected sum  $\#k(S^2 \times S^3)$  for all k (B-,Galicki,Nakamaye, Kollár). There are many distinct infinite series solutions giving infinitely many Sasaki-Seifert structures with SE metrics on the same manifold which can also depend on arbitrary parameters which implies a local moduli space.

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- On S<sup>5</sup> there are at least 82 distinct Sasaki-Seifert structures with Sasaki-Einstein metrics many of which occur with moduli (B-,Galicki,Kollár; Ghigi,Kollár; Li,Sun). The largest local moduli has real dimension 20 (B-,Macarini,van Koert). Moreover, there are at least 76 components of the Sasaki-Einstein moduli space (B-,Macarini,van Koert). These are distinguished by SH<sup>+,S<sup>1</sup></sup> and its mean Euler characteristic.

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- Many rational homology 5-spheres that satisfy the above torsion constraint admit Sasaki-Einstein metrics, but complete classification is still lacking (B-,Galicki; Kollár; B-,Nakamaye). Some admit infinitely many Sasaki-Seifert structures and some have arbitrary parameters implying moduli. Interesting results of Kollár show that there is a unique Sasaki-Seifert structure on 2M<sub>5</sub> whose moduli of SE structures is the moduli of genus two curves. Similarly 4M<sub>3</sub> has a unique Sasaki-Seifert structure with moduli of SE structures equal to the moduli of hyperelliptic genus four curves.

- If a simply connected 5-manifold admits a **Sasaki-Einstein metric**, it must be a **Smale** manifold  $M^5$  that is either one of the following or connected sums of the following:  $M_{\infty} = S^2 \times S^3$ ,  $M_1 = S^5$ ,  $M_j$  which satisfies  $H_2(M_j, \mathbb{Z}) = \mathbb{Z}_j \oplus \mathbb{Z}_j$ ,  $M^5 = M_{k_1} \# \cdots \# M_{k_s}$ ,  $k_i$  divides  $k_{i+1}$  or  $k_{i+1} = \infty$ .
- Not every Smale manifold can admit a **Sasaki-Einstein metric**. Then  $H_2(M, \mathbb{Z})_{tor}$  must be trivial or one of the following groups:  $\mathbb{Z}_m^2$ ,  $\mathbb{Z}_3^2$ ,  $\mathbb{Z}_3^4$ ,  $\mathbb{Z}_3^6$ ,  $\mathbb{Z}_3^8$ ,  $\mathbb{Z}_4^4$ ,  $\mathbb{Z}_5^4$  with n > 1 and  $m \ge 2$ . (Kollár)
- In dimension five there are Sasaki-Einstein metrics on the connected sum  $\#k(S^2 \times S^3)$  for all k (B-,Galicki,Nakamaye, Kollár). There are many distinct infinite series solutions giving infinitely many Sasaki-Seifert structures with SE metrics on the same manifold which can also depend on arbitrary parameters which implies a local moduli space.
- On S<sup>5</sup> there are at least 82 distinct Sasaki-Seifert structures with Sasaki-Einstein metrics many of which occur with moduli (B-,Galicki,Kollár; Ghigi,Kollár; Li,Sun). The largest local moduli has real dimension 20 (B-,Macarini,van Koert). Moreover, there are at least 76 components of the Sasaki-Einstein moduli space (B-,Macarini,van Koert). These are distinguished by SH+,S¹ and its mean Euler characteristic.
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- There are also many Sasaki-Einstein metrics on mixed Smale manifolds of the form kM∞ #nMm (B-,Galicki; Kollár; B-,Nakamaye) of course satisfying the torsion constraints.

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- In dimension 7 there are precisely 28 oriented diffemorphism types group bP<sub>8</sub>. Each such diffeomorphism type admits hundreds of Sasaki-Seifert structures each with a Sasaki-Einstein metric many of which have moduli (B-,Galicki,Kollár; Ghigi,Kollár).

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- Example: the sequence a = (2,3,7,43,43⋅39) gives a 2(44+4-5) = 86 real parameter family of Sasaki-Einstein metrics on the exotic sphere Σ<sub>6</sub> ∈ bP<sub>8</sub> i.e. 6 times the Milnor generator.

• The above sequence arises from a *boundary sequence* called **Sylvester's sequence** determined by the recursion relation  $c_{k+1} = 1 + c_1 \cdots c_k = c_k^2 - c_k + 1$  beginning with  $c_1 = 2$ . It starts as 2, 3, 7, 43, 1807, 3263443, 10650056950807, ... The importance of this sequence is that it satisfies  $\sum_{i=0}^{n} \frac{1}{c_i} = 1 - \frac{1}{c_0 \cdots c_n}$ . This sequence does not satisfy the **klt conditions**. So we choose numbers that are less than the last number of the sequence, so it will satisfy a **klt** condition and also certain gcd conditions. There are many such sequences, so

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- However, a Sasaki-Einstein metric is unique in the space  $\mathfrak{F}(\xi)$  up to a transverse holomorphic transformation (Nitta,Sekiya).

### LECTURE 3: Join Construction and Constant Scalar Curvature

• Join Revisited; the  $S_{\mathbf{w}}^3$ -Join

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- Fundamental Theorem: Geometry

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- The Einstein Hilbert Functional (E-H) and K-stability
- Stability Theorems continued
- Critical Points of E-H Functional
- Convexity of E-H Functional

### Recall the join construction

• We take  $M_1 = M$  to be a regular Sasaki manifold and  $M_2 = S_{\mathbf{w}}^3$ , the three dimensional weighted sphere. That is the weighted Sasakian structure  $S_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$  mentioned earlier with  $\mathbf{w} = (w_1, w_2)$ .

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- We take N to be a K\u00e4hler manifold with a constant scalar curvature metric. It is well known that \u00bbP\u00a4 [\u00bb] as an extremal K\u00e4hler metric. It follows that N × \u00bbP\u00a4 [\u00bb] has an extremal K\u00e4hler metric; hence, M \*h, \u00bb S\u00a4 \u00bb N\u00a4 has an extremal Sasaki metric.
- There is a 2-dimensional subcone  $\mathfrak{t}_{\mathbf{w}}^+$  of the Sasaki cone  $\mathfrak{t}^+$  of  $M \star_{h_1,h_2} S_{\mathbf{w}}^3$ .

# Join Revisited: the $S_{\mathbf{w}}^3$ -Join

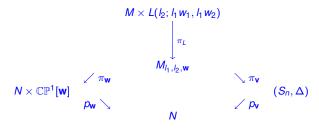
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- Our main object of study is the join  $M \star_{l_1, l_2} S_{\mathbf{w}}^3$  which is a smooth manifold if  $\gcd(l_2, l_1 w_i) = 1$ .
- An  $S^1$  orbibundle  $M \star_{l_1, l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$ , where N is compact Kähler.
- We take N to be a K\u00e4hler manifold with a constant scalar curvature metric. It is well known that \u00bbP\u00a4 [\u00bb] as an extremal K\u00e4hler metric. It follows that N × \u00bbP\u00a4 [\u00bb] has an extremal K\u00e4hler metric; hence, M \*h, \u00bb S\u00a4 \u00bb N\u00a4 has an extremal Sasaki metric.
- There is a 2-dimensional subcone  $\mathfrak{t}_{\mathbf{w}}^+$  of the Sasaki cone  $\mathfrak{t}^+$  of  $M \star_{l_1, l_2} S_{\mathbf{w}}^3$ .
- The join M \*h<sub>1,l<sub>2</sub></sub> S<sup>3</sup><sub>w</sub> can also be realized as a lens space bundle over N with fiber the lens space L(l<sub>2</sub>; l<sub>1</sub>w<sub>1</sub>, l<sub>1</sub>w<sub>2</sub>).

# Deforming the Reeb vector field in $\mathfrak{t}_{\mathbf{w}}^+$

We deform the Reeb vector field in t<sup>+</sup><sub>w</sub> to a new Reeb field ξ<sub>v</sub> to look for cscS and other extremal Sasaki metrics. Here v = (v<sub>1</sub>, v<sub>2</sub>) is real valued vector.

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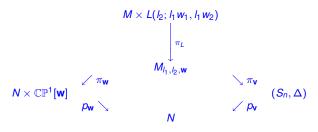
- We deform the Reeb vector field in  $\mathfrak{t}_{\mathbf{v}}^+$  to a new Reeb field  $\xi_{\mathbf{v}}$  to look for **cscS** and other **extremal Sasaki metrics**. Here  $\mathbf{v} = (v_1, v_2)$  is real valued vector.
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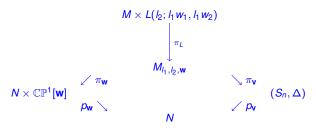


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• Geometry: Existence of extremal and CSC Sasaki metrics by deforming in the Sasaki cone

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Let  $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S_{\mathbf{w}}^3$  be the  $S_{\mathbf{w}}^3$ -join with a regular Sasaki manifold M which is an  $S^1$ -bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector  $\mathbf{w} = (w_1,w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$  with relatively prime components satisfying  $w_1 > w_2$  there exists a Reeb vector field  $\xi_{\mathbf{v}}$  in a 2-dimensional sub cone, the  $\mathbf{w}$ -cone, of the Sasaki cone on  $M_{l_1,l_2,\mathbf{w}}$  such that the corresponding ray of Sasakian structures  $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a_{l_1\mathbf{v}}, \Phi, g_a)$  has constant scalar curvature.

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  - Relation to CR Yamabe Problem (Jerison and Lee): For a Sasaki structure the Webster pseudo-Hermitian metric coincides with the transverse Kähler metric. So a CSC Sasaki metric provides a solution to the CR Yamabe Problem. It is know that when the CR Yamabe invariant  $\lambda(M)$  is nonpositive, the CSC metric is unique. However, when  $\lambda(M) > 0$  there can be several CSC solutions. Our results provides many such examples.

• The existence of an extra Hamiltonian Killing vector field from  $S_{\mathbf{w}}^3$  gives the 2-dimensional Sasaki w-cone  $\mathbf{t}_{\mathbf{w}}^+$ .

- The existence of an extra Hamiltonian Killing vector field from S<sup>3</sup><sub>w</sub> gives the 2-dimensional Sasaki w-cone t<sup>+</sup><sub>w</sub>.
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## Outline of proof of Geometry Theorem:

- The existence of an extra Hamiltonian Killing vector field from S<sup>3</sup><sub>w</sub> gives the 2-dimensional Sasaki w-cone t<sup>+</sup><sub>w</sub>.
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- The existence of multiple rays of CSC Sasaki metrics comes from a sign changing count.

## Fundamental Theorem: Topology

Topology: given the cohomology ring of M, the cohomology ring of M<sub>l1</sub>, l2, w can be determined in principal.

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#### Theorem (B-,Tønnesen-Friedman)

In each odd dimension 2p+3>5 there exist countably infinite simply connected toric contact manifolds  $M_{l_1,l_2,\mathbf{w}}$  of Reeb type depending on 4 positive integers  $l_1,l_2,w_1,w_2$  satisfying  $\gcd(l_2,l_1w_i)=\gcd(w_1,w_2)=1$ , and with integral cohomology ring

$$H^*(M_{l_1,l_2,\mathbf{w}},\mathbb{Z}) \approx \mathbb{Z}[x,y]/(w_1w_2l_1^2x^2,x^{p+1},x^2y,y^2)$$

where x,y are classes of degree 2 and 2p+1, respectively. Furthermore, with  $l_1, w_1, w_2$  fixed there are a finite number of diffeomorphism types with the given cohomology ring. Hence, in each such dimension there exist simply connected smooth manifolds with countably infinite toric contact structures of Reeb type that are inequivalent as contact structures.

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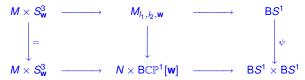
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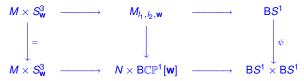
 In special cases we can determine the diffeomorphism (homeomorphism, homotopy) types.

From join construction get commutative diagram of fibrations:



where BG is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.

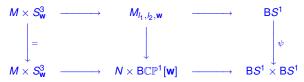
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• Given the differentials in the spectral sequence of the fibration  $M \longrightarrow N \longrightarrow BS^1$ , use the commutative diagram to compute the cohomology ring of the contact manifold  $M_{l_1,b,\mathbf{w}}$  with

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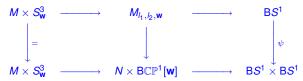


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- Given the differentials in the spectral sequence of the fibration M→N→N→BS¹, use the
  commutative diagram to compute the cohomology ring of the contact manifold M<sub>l1,l2,w</sub> with
- orbifold cohomology:

$$H^r_{orb}(\mathbb{CP}^1[\mathbf{w}], \mathbb{Z}) = H^r(\mathsf{BCP}^1[\mathbf{w}], \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } r = 0, 2, \\ \mathbb{Z}_{w_1 w_2} & \text{for } r > 2 \text{ even,} \\ 0 & \text{for } r \text{ odd.} \end{cases}$$

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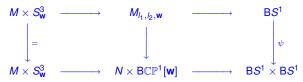
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- For Theorem (2) we take the sphere  $M = S^{2p+1}$  with p > 1 in which case the cohomology ring follows by computing the differentials. p = 1 case later ( $S^3$ -bundles over  $S^2$ ).
- The finiteness of the diffeomorphism types follows by Sullivan's rational homotopy theory.  $M_{l_1,l_2,\mathbf{w}}$  is formal.

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- Some of the same type of results have been obtained on 5-manifolds whose fundamental group is a non-Abelian extension of  $\pi_1(\Sigma_q)$  in Castañeda's thesis.

• The contact structure  $\mathcal{D}_{l_1,1,\mathbf{w}}$  on  $S^3 \times \Sigma_g$  has a **4-bouquet** for all genera g.

- The contact structure  $\mathcal{D}_{h,1,\mathbf{w}}$  on  $S^3 \times \Sigma_g$  has a **4-bouquet** for all genera g.
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- If we take  $l_2 > 1$  with  $gcd(l_2, 210)$  and g = 0 we get a 3-bouquet on  $S^3 \times S^2$ . The m = 0 term doesn't occur.
- If, however, g>0 with  $l_2>1$  the topology of the join is not completely known, so we can say nothing about bouquets.

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$$Fut(Y_0,\xi,a) := \frac{\mathbf{V}_{\xi}}{n} D_a \left( \frac{\mathbf{S}_{\xi}}{\mathbf{V}_{\xi}} \right) + \frac{\mathbf{S}_{\xi} D_a \mathbf{V}_{\xi}}{n(n+1) \mathbf{V}_{\xi}}$$

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We say that (Y, ξ) is K-semistable if for each T<sub>C</sub> with ξ ∈ t and any T<sub>C</sub>-equivariant test configuration Fut(Y<sub>0</sub>, ξ, a) ≥ 0. It is K-polystable if equality holds only if it is a product configuration, that is, Y<sub>0</sub> is isomorphic to Y.

#### Theorem (1)

The set of critical points of the Einstein–Hilbert functional  $H_{\xi}$  is the union of the zeros of the Sasaki-Futaki invariant and of the total transversal scalar curvature. In particular, if a Reeb vector field admits a compatible cscS metric then it is a critical point of the Einstein–Hilbert functional.

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If  $(Y, \xi)$  is K-semistable then the Sasaki–Futaki invariant  $\mathbf{F}_{\xi}$  vanishes identically, and  $\xi$  is a critical point of the Einstein-Hilbert functional  $\mathbf{H}(\xi)$ . Alternatively, if  $\xi$  is not a critical point of  $\mathbf{H}(\xi)$  then  $(Y, \xi)$  is not K-semistable.

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- Theorem then follows from Equation (1).

## Theorem (3)

Suppose the Sasaki cone is exhausted by extremal Sasaki metrics and that the total transverse scalar curvature does not vanish. Then the set of critical points of the Einstein-Hilbert functional is precisely the set of rays in the Sasaki cone with constant scalar curvature. In particular, in this case a Sasakian structure  $\mathcal{S}=(\xi,\eta,\Phi,g)$  has constant scalar curvature if and only if  $(Y,\xi)$  is K-semistable.

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Let M be a regular Sasaki manifold with constant transverse scalar curvature, and consider the  $S^{\mathbf{w}}_{\mathbf{y}}$ -join  $M_{l_1,l_2,\mathbf{w}}$ . Its  $\mathbf{w}$ -cone  $\mathbf{t}^{\mathbf{t}}_{\mathbf{w}}$  has a cscS ray  $\mathbf{t}_{\xi}$  if and only if the Sasaki-Futaki invariant  $\mathbf{F}_{\xi}$  vanishes on the Lie algebra  $\mathbf{t}_{\mathbf{w}} \otimes \mathbb{C}$ . Then for  $\xi \in \mathbf{t}^{\mathbf{t}}_{\mathbf{w}}$  the polarized affine cone  $(Y,\xi)$  associated to  $M_{h_1,h_2,\mathbf{w}}$  is K-semistable if and only if the Sasakian structure  $\mathcal{S}=(\xi,\eta,\Phi,g)$  on  $M_{h_1,h_2,\mathbf{w}}$  has constant scalar curvature (up to isotopy).

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Let M be a regular Sasaki manifold with constant transverse scalar curvature, and consider the  $S^3_{\mathbf{w}}$ -join  $M_{l_1,l_2,\mathbf{w}}$ . Its  $\mathbf{w}$ -cone  $\mathbf{t}^+_{\mathbf{w}}$  has a cscS ray  $\mathbf{r}_{\xi}$  if and only if the Sasaki-Futaki invariant  $\mathbf{F}_{\xi}$  vanishes on the Lie algebra  $\mathbf{t}_{\mathbf{w}} \otimes \mathbb{C}$ . Then for  $\xi \in \mathbf{t}^+_{\mathbf{w}}$  the polarized affine cone  $(Y,\xi)$  associated to  $M_{l_1,l_2,\mathbf{w}}$  is K-semistable if and only if the Sasakian structure  $\mathcal{S}=(\xi,\eta,\Phi,g)$  on  $M_{l_1,l_2,\mathbf{w}}$  has constant scalar curvature (up to isotopy).

Outline of Proof: This involves a long calculation of the admissible Kähler construction together with the S<sup>3</sup><sub>w</sub>-Sasaki join construction which expresses the Sasaki-Futaki invariant in terms of certain polynomial on the Sasaki cone.

## Theorem (5)

Assume that the total transversal scalar curvature is sign definite and bounded away from 0 on each transversal set of the Sasaki cone, then there exists at least one Reeb vector field for which the Sasaki–Futaki invariant  $\mathfrak{S}$  vanishes identically.

## Theorem (5)

Assume that the total transversal scalar curvature is sign definite and bounded away from 0 on each transversal set of the Sasaki cone, then there exists at least one Reeb vector field for which the Sasaki–Futaki invariant ST vanishes identically.

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- A result of He, Sun says that  $V_{\xi}$  is strictly convex on  $\Sigma$  and tends to infinity on the boundary  $\partial \Sigma$ .
- So  $\mathbf{H}(\xi)$  must reach a minimum somewhere in  $\Sigma$  and the variational formula implies  $\mathfrak{S}\mathfrak{F}=0$ .

## Theorem (6)

Let  $(M,D,J,g,\xi)$  be either a cscS compact manifold of negative transverse scalar curvature or a compact Sasaki- $\eta$ -Einstein manifold of positive transverse scalar curvature. Then H is transversally convex at  $\xi$  (or transversally concave in the negative cscS case with n odd).

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and the well known inequality  $\|d(\eta(a))\|_g^2 \ge \lambda_1 \|\eta(a)\|_g^2$ .

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 Proof of Theorem (6): For negative transverse scalar curvature, the theorem follows immediately from (1) of the lemma.

#### Theorem (6)

Let  $(M, D, J, g, \xi)$  be either a cscS compact manifold of negative transverse scalar curvature or a compact Sasaki-η-Einstein manifold of positive transverse scalar curvature. Then H is transversally convex at  $\xi$  (or transversally concave in the negative cscS case with n odd).

- Lemma: Let  $(M^{2n+1}, D, J, g, \xi)$  be a cscS compact manifold of non-zero transversal scalar curvature, and  $T \subset \mathfrak{CR}(D, J) \cap \mathsf{Isom}(g)$  be the maximal compact torus with Lie algebra  $\mathfrak{t}$ .
  - ① If the transverse scalar curvature is negative, the E-H functional H is convex if n is even and concave if n is odd.
- 2 If the transverse scalar curvature is positive and the first non-zero eigenvalue  $\lambda_1$  of the Laplacian, restricted to the space of T-invariant functions is bounded below by  $\frac{s_g^i}{2g+1}$ , then **H** is convex near  $\xi$ . Moreover, in both cases the cscS ray is isolated in the Sasaki cone.
- The lemma follows from the second variational formula at a cscS metric:

$$\frac{d^2}{dt^2}_{t=0}\mathbf{H}(\xi_t) = n(n+1)(2n+1)(s_g^T)^n \left( \|d(\eta(a))\|_g^2 - \frac{s_g^T}{(2n+1)} \|\eta(a)\|_g^2 \right)$$

and the well known inequality  $\|d(\eta(a))\|_q^2 \ge \lambda_1 \|\eta(a)\|_q^2$ .

- Proof of Theorem (6): For negative transverse scalar curvature, the theorem follows immediately from (1) of the lemma.
- For the Sasaki- $\eta$ -Einstein case a bound  $\lambda_1 \geq \frac{s_2^T}{2n}$  due to Matsushima together with (2) of the lemma gives the result.

#### Theorem (6)

Let  $(M, D, J, g, \xi)$  be either a cscS compact manifold of negative transverse scalar curvature or a compact Sasaki- $\eta$ -Einstein manifold of positive transverse scalar curvature. Then H is transversally convex at  $\xi$  (or transversally concave in the negative cscS case with n odd).

- Lemma: Let  $(M^{2n+1}, D, J, g, \xi)$  be a cscS compact manifold of non-zero transversal scalar curvature, and  $T \subset \mathfrak{CR}(D, J) \cap \mathsf{Isom}(g)$  be the maximal compact torus with Lie algebra  $\mathfrak{t}$ .
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- Moreover, in both cases the cscS ray is isolated in the Sasaki cone.
- The lemma follows from the second variational formula at a cscS metric:

$$\frac{d^2}{dt^2}_{t=0}\mathbf{H}(\xi_t) = n(n+1)(2n+1)(s_g^T)^n \left( \|d(\eta(a))\|_g^2 - \frac{s_g^T}{(2n+1)} \|\eta(a)\|_g^2 \right)$$

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- Proof of Theorem (6): For negative transverse scalar curvature, the theorem follows immediately from (1) of the lemma.
- For the Sasaki- $\eta$ -Einstein case a bound  $\lambda_1 \geq \frac{s_2^T}{2n}$  due to Matsushima together with (2) of the lemma gives the result.
- This lemma leads to the general question: Are the rays of constant scalar curvature Sasaki metrics isolated in the Sasaki cone?

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