EXTREMAL SASAKIAN GEOMETRY

Charles Boyer

University of New Mexico

October 28, 2015
The 21th Symposium on Complex Geometry,
Kanazawa, Japan

Dedicated to the memory of Professor Shigeo Sasaki

Outline of Talk

1. The Classical Period – a Brief History

Outline of Talk

1. The Classical Period – a Brief History
2. The Foundations: 1. Contact Geometry

Outline of Talk

1. The Classical Period – a Brief History
2. The Foundations: 1. Contact Geometry
3. The Foundations: 2. Sasakian Geometry

Outline of Talk

1. The Classical Period – a Brief History
2. The Foundations: 1. Contact Geometry
3. The Foundations: 2. Sasakian Geometry
4. Sasaki Cones and Affine Cones

Outline of Talk

1. The Classical Period – a Brief History
2. The Foundations: 1. Contact Geometry
3. The Foundations: 2. Sasakian Geometry
4. Sasaki Cones and Affine Cones
5. Deformations of Sasakian Structures

Outline of Talk

Outline of Talk

1. The Classical Period – a Brief History
2. The Foundations: 1. Contact Geometry
3. The Foundations: 2. Sasakian Geometry
4. Sasaki Cones and Affine Cones
5. Deformations of Sasakian Structures
6. Transverse Holomorphic Transformations
7. Extremal Sasaki Metrics

Outline of Talk

Outline of Talk

Outline of Talk

1. The Classical Period – a Brief History
2. The Foundations: 1. Contact Geometry
3. The Foundations: 2. Sasakian Geometry
4. Sasaki Cones and Affine Cones
5. Deformations of Sasakian Structures
6. Transverse Holomorphic Transformations
7. Extremal Sasaki Metrics
8. The Sasaki-Eintein Problem: Some History
9. The $S^3_{\mathbf{w}}$-Join Construction
10. The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions

Outline of Talk

1. The Classical Period – a Brief History
2. The Foundations: 1. Contact Geometry
3. The Foundations: 2. Sasakian Geometry
4. Sasaki Cones and Affine Cones
5. Deformations of Sasakian Structures
6. Transverse Holomorphic Transformations
7. Extremal Sasaki Metrics
8. The Sasaki-Eintein Problem: Some History
9. The $S^3_{\mathbf{w}}$-Join Construction
10. The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions
11. The Einstein-Hilbert Functional

Outline of Talk

Outline of Talk

1. The Classical Period – a Brief History
2. The Foundations: 1. Contact Geometry
3. The Foundations: 2. Sasakian Geometry
4. Sasaki Cones and Affine Cones
5. Deformations of Sasakian Structures
6. Transverse Holomorphic Transformations
7. Extremal Sasaki Metrics
8. The Sasaki-Eintein Problem: Some History
9. The $S^3_{\mathbf{w}}$-Join Construction
10. The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions
11. The Einstein-Hilbert Functional
12. Sasakian K-Stability
13. The Einstein-Hilbert Functional and K-Stability

Outline of Talk

Outline of Talk

Outline of Talk

Outline of Talk

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow$ (ϕ, ξ, η) structures $\longrightarrow$ (almost contact)

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow$ (ϕ, ξ, η) structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow (\phi, \xi, \eta)$ structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow (\phi,\xi,\eta)$ structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow$ (ϕ, ξ, η) structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow$ (ϕ, ξ, η) structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**
7. Hatakeyama (1963): Relation of regular normal contact structures to the Boothby-Wang construction.

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow$ (ϕ, ξ, η) structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**
7. Hatakeyama (1963): Relation of regular normal contact structures to the Boothby-Wang construction.
8. Sasaki wrote Lecture Notes in 3 parts entitled "**Almost Contact Manifolds**", part 1 (1965), part 2 (1967), part 3 (1968). Unfortunately, these notes were never published and are difficult to obtain in the west.

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow (\phi, \xi, \eta)$ structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**
7. Hatakeyama (1963): Relation of regular normal contact structures to the Boothby-Wang construction.
8. Sasaki wrote Lecture Notes in 3 parts entitled "**Almost Contact Manifolds**", part 1 (1965), part 2 (1967), part 3 (1968). Unfortunately, these notes were never published and are difficult to obtain in the west.
9. The earliest reference referring to a normal contact metric space as a **Sasakian space** is a 1965 paper of S. Tachibana entitled "**On Harmonic Tensors in Compact Sasakian Spaces**".

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960): $\longrightarrow (\phi, \xi, \eta)$ structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**
7. Hatakeyama (1963): Relation of regular normal contact structures to the Boothby-Wang construction.
8. Sasaki wrote Lecture Notes in 3 parts entitled "**Almost Contact Manifolds**", part 1 (1965), part 2 (1967), part 3 (1968). Unfortunately, these notes were never published and are difficult to obtain in the west.
9. The earliest reference referring to a normal contact metric space as a **Sasakian space** is a 1965 paper of S. Tachibana entitled "**On Harmonic Tensors in Compact Sasakian Spaces**".
10. Subsequently, almost contact structures, and **Sasakian** structures were ardently developed by Japaneese mathematicians during the 1960's and 1970's, particularly by S. Tanno.

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow$ (ϕ, ξ, η) structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**
7. Hatakeyama (1963): Relation of regular normal contact structures to the Boothby-Wang construction.
8. Sasaki wrote Lecture Notes in 3 parts entitled "**Almost Contact Manifolds**", part 1 (1965), part 2 (1967), part 3 (1968). Unfortunately, these notes were never published and are difficult to obtain in the west.
9. The earliest reference referring to a normal contact metric space as a **Sasakian space** is a 1965 paper of S. Tachibana entitled "**On Harmonic Tensors in Compact Sasakian Spaces**".
10. Subsequently, almost contact structures, and **Sasakian** structures were ardently developed by Japaneese mathematicians during the 1960's and 1970's, particularly by S. Tanno.
11. Relation to Brieskorn manifolds: Abe, Takahashi, Sasaki-Hsu

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow$ (ϕ, ξ, η) structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**
7. Hatakeyama (1963): Relation of regular normal contact structures to the Boothby-Wang construction.
8. Sasaki wrote Lecture Notes in 3 parts entitled "**Almost Contact Manifolds**", part 1 (1965), part 2 (1967), part 3 (1968). Unfortunately, these notes were never published and are difficult to obtain in the west.
9. The earliest reference referring to a normal contact metric space as a **Sasakian space** is a 1965 paper of S. Tachibana entitled "**On Harmonic Tensors in Compact Sasakian Spaces**".
10. Subsequently, almost contact structures, and **Sasakian** structures were ardently developed by Japaneese mathematicians during the 1960's and 1970's, particularly by S. Tanno.
11. Relation to Brieskorn manifolds: Abe, Takahashi, Sasaki-Hsu
12. Sasakian 3-structures: introduced independently by Udriste, Kuo and developed further by Ishihara, Kashiwada, Konishi, Tachibana, Tanno, Yu.

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960):$\longrightarrow$ (ϕ, ξ, η) structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**
7. Hatakeyama (1963): Relation of regular normal contact structures to the Boothby-Wang construction.
8. Sasaki wrote Lecture Notes in 3 parts entitled "**Almost Contact Manifolds**", part 1 (1965), part 2 (1967), part 3 (1968). Unfortunately, these notes were never published and are difficult to obtain in the west.
9. The earliest reference referring to a normal contact metric space as a **Sasakian space** is a 1965 paper of S. Tachibana entitled "**On Harmonic Tensors in Compact Sasakian Spaces**".
10. Subsequently, almost contact structures, and **Sasakian** structures were ardently developed by Japaneese mathematicians during the 1960's and 1970's, particularly by S. Tanno.
11. Relation to Brieskorn manifolds: Abe, Takahashi, Sasaki-Hsu
12. Sasakian 3-structures: introduced independently by Udriste, Kuo and developed further by Ishihara, Kashiwada, Konishi, Tachibana, Tanno, Yu.
13. Some books: Blair (1976), Yano-Kon (1984)

Sasaki Geometry: the Classical Period (1960–1980) – a Brief History

1. Boothby-Wang (1958); Gray(1959): $\longrightarrow$ Topology of (almost) Contact structures
2. Sasaki (1960): $\longrightarrow (\phi, \xi, \eta)$ structures $\longrightarrow$ (almost contact)
3. Sasaki, Hatakeyama (1961): Normal almost contact (metric) structures
4. Sasaki, Hatakeyama (1962): Normal contact metric structures
5. Okumura (1962): Some relations of normal contact metric structures with Einstein metrics.
6. Hatakeyama, Ogawa, Tanno (1963): Contact metric structure + ξ a Killing vector $\longrightarrow$ **K-contact**
7. Hatakeyama (1963): Relation of regular normal contact structures to the Boothby-Wang construction.
8. Sasaki wrote Lecture Notes in 3 parts entitled "**Almost Contact Manifolds**", part 1 (1965), part 2 (1967), part 3 (1968). Unfortunately, these notes were never published and are difficult to obtain in the west.
9. The earliest reference referring to a normal contact metric space as a **Sasakian space** is a 1965 paper of S. Tachibana entitled "**On Harmonic Tensors in Compact Sasakian Spaces**".
10. Subsequently, almost contact structures, and **Sasakian** structures were ardently developed by Japaneese mathematicians during the 1960's and 1970's, particularly by S. Tanno.
11. Relation to Brieskorn manifolds: Abe, Takahashi, Sasaki-Hsu
12. Sasakian 3-structures: introduced independently by Udriste, Kuo and developed further by Ishihara, Kashiwada, Konishi, Tachibana, Tanno, Yu.
13. Some books: Blair (1976), Yano-Kon (1984)
14. I apologize if I inadvertently left some names out.

The Foundations: 1. Contact Geometry

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.

The Foundations: 1. Contact Geometry

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.
2. A **contact 1-form** η such that

$$\eta \wedge (d\eta)^n \neq 0.$$

The Foundations: 1. Contact Geometry

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.
2. A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
3. defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So $\{$oriented contact 1-forms in $\mathcal{D}\} \approx C^\infty(M)^+$

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.
2. A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
3. defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So $\{$oriented contact 1-forms in $\mathcal{D}\} \approx C^\infty(M)^+$
4. The pair $(M, \mathcal{D})$ is called a **contact manifold**.

The Foundations: 1. Contact Geometry

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.
2. A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
3. defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So $\{$oriented contact 1-forms in $\mathcal{D}\} \approx C^\infty(M)^+$
4. The pair $(M, \mathcal{D})$ is called a **contact manifold**.
5. If we choose a contact 1-form η, there is a unique vector field ξ, called the **Reeb vector field**, satisfying
$$\eta(\xi) = 1, \qquad \xi \rfloor d\eta = 0.$$

The Foundations: 1. Contact Geometry

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.
2. A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
3. defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So $\{$oriented contact 1-forms in $\mathcal{D}\} \approx C^\infty(M)^+$
4. The pair $(M, \mathcal{D})$ is called a **contact manifold**.
5. If we choose a contact 1-form η, there is a unique vector field ξ, called the **Reeb vector field**, satisfying
$$\eta(\xi) = 1, \qquad \xi \rfloor d\eta = 0.$$
6. The **characteristic foliation** $\mathcal{F}_\xi$ is the 1-dim'l foliation defined by ξ: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k = 1$; otherwise, it is **irregular**. We also say that the **contact form** η is **quasi-regular, regular, irregular**.

The Foundations: 1. Contact Geometry

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.
2. A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
3. defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So $\{$oriented contact 1-forms in $\mathcal{D}\} \approx C^\infty(M)^+$
4. The pair $(M, \mathcal{D})$ is called a **contact manifold**.
5. If we choose a contact 1-form η, there is a unique vector field ξ, called the **Reeb vector field**, satisfying
$$\eta(\xi) = 1, \qquad \xi \rfloor d\eta = 0.$$
6. The **characteristic foliation** $\mathcal{F}_\xi$ is the 1-dim'l foliation defined by ξ: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k=1$; otherwise, it is **irregular**. We also say that the **contact form** η is **quasi-regular, regular, irregular**.
7. Most contact forms in a contact structure $\mathcal{D}$ are **irregular**

The Foundations: 1. Contact Geometry

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.
2. A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
3. defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So $\{$oriented contact 1-forms in $\mathcal{D}\} \approx C^\infty(M)^+$
4. The pair $(M, \mathcal{D})$ is called a **contact manifold**.
5. If we choose a contact 1-form η, there is a unique vector field ξ, called the **Reeb vector field**, satisfying
$$\eta(\xi) = 1, \qquad \xi \rfloor d\eta = 0.$$
6. The **characteristic foliation** $\mathcal{F}_\xi$ is the 1-dim'l foliation defined by ξ: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k = 1$; otherwise, it is **irregular**. We also say that the **contact form** η is **quasi-regular, regular, irregular**.
7. Most contact forms in a contact structure $\mathcal{D}$ are **irregular**
8. We can choose a **compatible almost complex structure** J on $\mathcal{D}$, that is one that satisfies the two conditions
$$d\eta(JX, JY) = d\eta(X, Y) \qquad d\eta(JX, Y) > 0$$
for any sections X, Y of $\mathcal{D}$.

The Foundations: 1. Contact Geometry

1. A **Closed Manifold** M of dimension $2n+1$, i.e. compact without boundary.
2. A **contact 1-form** η such that
$$\eta\wedge(d\eta)^n\neq 0.$$
3. defines a **contact structure**
$$\eta'\sim\eta \iff \eta'=f\eta$$
for some $f\neq 0$, take $f>0$, or equivalently a codimension 1 subbundle $\mathcal{D}=\mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure. So $\{$oriented contact 1-forms in $\mathcal{D}\}\approx C^\infty(M)^+$
4. The pair $(M,\mathcal{D})$ is called a **contact manifold**.
5. If we choose a contact 1-form η, there is a unique vector field ξ, called the **Reeb vector field**, satisfying
$$\eta(\xi)=1,\qquad \xi\rfloor d\eta=0.$$
6. The **characteristic foliation** $\mathcal{F}_\xi$ is the 1-dim'l foliation defined by ξ: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k=1$; otherwise, it is **irregular**. We also say that the **contact form** η is **quasi-regular, regular, irregular**.
7. Most contact forms in a contact structure $\mathcal{D}$ are **irregular**
8. We can choose a **compatible almost complex structure** J on $\mathcal{D}$, that is one that satisfies the two conditions
$$d\eta(JX,JY)=d\eta(X,Y)\qquad d\eta(JX,Y)>0$$
for any sections X,Y of $\mathcal{D}$.
9. The **almost complex structure** J extends to an endomorphism Φ of TM satisfying $\Phi\xi=0$.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes 1\!\!1) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.

- There is a 'canonical' compatible metric $g = d\eta\circ(\Phi\otimes 1\!\!1) + \eta\otimes\eta$. Quadruple $\mathcal{S} = (\xi,\eta,\Phi,g)$ called **contact metric structure**. Contact metric manifold $(M,\mathcal{S})$.
- $g_{\mathcal{D}} = d\eta\circ(\Phi\otimes 1\!\!1)$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.

The Foundations: 2. Sasakian Geometry

- There is a ‘canonical’ compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes \mathbb{1})$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes \mathbb{1})$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.

The Foundations: 2. Sasakian Geometry

- There is a ‘canonical’ compatible metric $g = d\eta\circ(\Phi\otimes 1\!\!1)+\eta\otimes\eta$. Quadruple $\mathcal{S}=(\xi,\eta,\Phi,g)$ called **contact metric structure**. Contact metric manifold $(M,\mathcal{S})$.
- $g_{\mathcal{D}} = d\eta\circ(\Phi\otimes 1\!\!1)$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D},J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX,Y)>0$ says that $(\mathcal{D},J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D},J)$ defines a **CR structure**.

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes \mathbb{1})$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D}, J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes \mathbb{1})$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D}, J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M, \mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M) = M \times \mathbb{R}^+, dr^2 + r^2 g)$ is Kähler.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta\circ(\Phi\otimes 1\!\!1)+\eta\otimes\eta$. Quadruple $\mathcal{S}=(\xi,\eta,\Phi,g)$ called **contact metric structure**. Contact metric manifold $(M,\mathcal{S})$.
- $g_{\mathcal{D}} = d\eta\circ(\Phi\otimes 1\!\!1)$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D},J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX,Y)>0$ says that $(\mathcal{D},J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D},J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ is **K-contact** if $\mathcal{L}_\xi g=0$ (or $\mathcal{L}_\xi\Phi=0$). It is **Sasakian** if in addition $(\mathcal{D},J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M,\mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M)=M\times\mathbb{R}^+, dr^2+r^2g)$ is Kähler.
- Some properties of a Sasaki manifold $(M,\mathcal{S})$ of dimension $2n+1$:

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes 1\!\!1) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes 1\!\!1)$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D}, J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M, \mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M) = M \times \mathbb{R}^+, dr^2 + r^2 g)$ is Kähler.
- Some properties of a Sasaki manifold $(M, \mathcal{S})$ of dimension $2n+1$:
  1. Any Sasaki structure $\mathcal{S}$ has at least an S^1 symmetry.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes \mathbb{1})$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D}, J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M, \mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M) = M \times \mathbb{R}^+, dr^2 + r^2 g)$ is Kähler.
- Some properties of a Sasaki manifold $(M, \mathcal{S})$ of dimension $2n + 1$:
  1. Any Sasaki structure $\mathcal{S}$ has at least an S^1 symmetry.
  2. The **characteristic foliation** $\mathcal{F}_\xi$ is Riemannian, that is, a **Riemannian flow**.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes \mathbb{1})$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D}, J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M, \mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M) = M \times \mathbb{R}^+, dr^2 + r^2 g)$ is Kähler.
- Some properties of a Sasaki manifold $(M, \mathcal{S})$ of dimension $2n+1$:
  1. Any Sasaki structure $\mathcal{S}$ has at least an S^1 symmetry.
  2. The **characteristic foliation** $\mathcal{F}_\xi$ is Riemannian, that is, a **Riemannian flow**.
  3. If $\mathcal{S}$ is irregular, then the closure $\bar{\mathcal{F}}_\xi$ is a **torus** T^k of dimension $1 \leq k \leq n+1$.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes 1\!\!1) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes 1\!\!1)$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D}, J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M, \mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M) = M \times \mathbb{R}^+, dr^2 + r^2 g)$ is Kähler.
- Some properties of a Sasaki manifold $(M, \mathcal{S})$ of dimension $2n+1$:
  1. Any Sasaki structure $\mathcal{S}$ has at least an S^1 symmetry.
  2. The **characteristic foliation** $\mathcal{F}_\xi$ is Riemannian, that is, a **Riemannian flow**.
  3. If $\mathcal{S}$ is irregular, then the closure $\bar{\mathcal{F}}_\xi$ is a **torus** T^k of dimension $1 \leq k \leq n+1$.
  4. The metric g is **bundle-like**.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes \mathbb{1})$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D}, J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M, \mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M) = M \times \mathbb{R}^+, dr^2 + r^2 g)$ is Kähler.
- Some properties of a Sasaki manifold $(M, \mathcal{S})$ of dimension $2n + 1$:
  1. Any Sasaki structure $\mathcal{S}$ has at least an S^1 symmetry.
  2. The **characteristic foliation** $\mathcal{F}_\xi$ is Riemannian, that is, a **Riemannian flow**.
  3. If $\mathcal{S}$ is irregular, then the closure $\bar{\mathcal{F}}_\xi$ is a **torus** T^k of dimension $1 \leq k \leq n + 1$.
  4. The metric g is **bundle-like**.
  5. The leaves of $\mathcal{F}_\xi$ (orbits of ξ) are **totally geodesic**.

The Foundations: 2. Sasakian Geometry

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
- $g_{\mathcal{D}} = d\eta \circ (\Phi \otimes \mathbb{1})$ defines a metric in $\mathcal{D}$ called the **transverse metric** and $\omega^T = d\eta$ is a **transverse symplectic form** in $\mathcal{D}$.
- The pair $(\mathcal{D}, J)$ defines an **almost CR structure** on $\mathcal{D}$ with $\Phi|_{\mathcal{D}} = J$.
- $d\eta$ is called the **Levi form** of $\mathcal{D}$ and the condition $d\eta(JX, Y) > 0$ says that $(\mathcal{D}, J)$ is **strictly pseudo-convex** abbreviated as **sψCR**.
- We are mainly interested in the case that the almost CR structure is **integrable**, that is, that $(\mathcal{D}, J)$ defines a **CR structure**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- $(M, \mathcal{S})$ is Sasaki $\iff$ the **metric cone** $(C(M) = M \times \mathbb{R}^+, dr^2 + r^2 g)$ is Kähler.
- Some properties of a Sasaki manifold $(M, \mathcal{S})$ of dimension $2n+1$:
  1. Any Sasaki structure $\mathcal{S}$ has at least an S^1 symmetry.
  2. The **characteristic foliation** $\mathcal{F}_\xi$ is Riemannian, that is, a **Riemannian flow**.
  3. If $\mathcal{S}$ is irregular, then the closure $\bar{\mathcal{F}}_\xi$ is a **torus** T^k of dimension $1 \leq k \leq n+1$.
  4. The metric g is **bundle-like**.
  5. The leaves of $\mathcal{F}_\xi$ (orbits of ξ) are **totally geodesic**.
  6. The **Ricci curvature** of g satisfies $\mathrm{Ric}_g(X, \xi) = 2n\eta(X)$ for any vector field X.

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M.
  Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
    1. $\mathfrak{t}_k$ the Lie algebra of T^k

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.
  6. The set of extremal rays $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.
  6. The set of extremal rays $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.
- **The Affine Cone**

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.
  6. The set of extremal rays $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.
- **The Affine Cone**
  1. The cone $C(M)$ is an **affine cone** which we denote by $Y = C(M)$

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.
  6. The set of extremal rays $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.
- **The Affine Cone**
  1. The cone $C(M)$ is an **affine cone** which we denote by $Y = C(M)$
  2. The **Kählerian structure** of the cone Y is determined by the **Sasakian structure** on M or equivalently by the Reeb vector field ξ

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.
  6. The set of extremal rays $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.
- **The Affine Cone**
  1. The cone $C(M)$ is an **affine cone** which we denote by $Y = C(M)$
  2. The **Kählerian structure** of the cone Y is determined by the **Sasakian structure** on M or equivalently by the Reeb vector field ξ
  3. A ξ-equivariant function $r : Y \longrightarrow \mathbb{R}^+$ gives the Kähler form $\Omega = i\partial\bar{\partial}r^2$

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.
  6. The set of extremal rays $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.
- **The Affine Cone**
  1. The cone $C(M)$ is an **affine cone** which we denote by $Y = C(M)$
  2. The **Kählerian structure** of the cone Y is determined by the **Sasakian structure** on M or equivalently by the Reeb vector field ξ
  3. A ξ-equivariant function $r : Y \longrightarrow \mathbb{R}^+$ gives the Kähler form $\Omega = i\partial\bar{\partial}r^2$
  4. ξ satisfies $\xi = I\Psi$ where $\Psi = r\frac{\partial}{\partial r}$, I is complex structure on Y.

The Sasaki Cone and the Affine cone

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.
  6. The set of extremal rays $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.
- **The Affine Cone**
  1. The cone $C(M)$ is an **affine cone** which we denote by $Y = C(M)$
  2. The **Kählerian structure** of the cone Y is determined by the **Sasakian structure** on M or equivalently by the Reeb vector field ξ
  3. A ξ-equivariant function $r : Y \longrightarrow \mathbb{R}^+$ gives the Kähler form $\Omega = i\partial\bar{\partial}r^2$
  4. ξ satisfies $\xi = I\Psi$ where $\Psi = r\frac{\partial}{\partial r}$, I is complex structure on Y.
  5. We can lift the **Sasaki cone** $\mathfrak{t}_k^+$ to Y, denoted also by $\mathfrak{t}_k^+$.

- Distinct **Sasakian structures** in the same underlying **CR structure** $\longrightarrow$ **Sasaki cone** on M. Distinct **Kähler forms** in the same underlying **affine cone** $\longrightarrow$ **Sasaki cone** on Y.
- On a compact Sasaki manifold $(M^{2n+1}, \mathcal{S})$ the Sasaki automophism group $\mathfrak{Aut}(\mathcal{S})$ contains a torus T^k of dimension $1 \leq k \leq n+1$. The case $k = n+1$ is a **toric Sasakian** structure.
- **Sasaki cone**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$.
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n+1$ and if $\dim \kappa(\mathcal{D}, J) = n+1$, M is **toric Sasakian**.
  6. The set of extremal rays $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.
- **The Affine Cone**
  1. The cone $C(M)$ is an **affine cone** which we denote by $Y = C(M)$
  2. The **Kählerian structure** of the cone Y is determined by the **Sasakian structure** on M or equivalently by the Reeb vector field ξ
  3. A ξ-equivariant function $r : Y \longrightarrow \mathbb{R}^+$ gives the Kähler form $\Omega = i\partial\bar{\partial}r^2$
  4. ξ satisfies $\xi = I\Psi$ where $\Psi = r\frac{\partial}{\partial r}$, I is complex structure on Y.
  5. We can lift the **Sasaki cone** $\mathfrak{t}_k^+$ to Y, denoted also by $\mathfrak{t}_k^+$.
  6. For each $\xi \in \mathfrak{t}_k^+$ have different **Kähler form** Ω on Y, get a **polarized affine variety** (Y, ξ).

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
- First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
- First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
    1. Generically after this type of deformation the transverse holonomy becomes irreducible.

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
- First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  1. Generically after this type of deformation the transverse holonomy becomes irreducible.
- Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
- First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  1. Generically after this type of deformation the transverse holonomy becomes irreducible.
- Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.
  1. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})$.

- Three Types of **Deformations** of Sasakian Structures
- First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  1. Generically after this type of deformation the transverse holonomy becomes irreducible.
- Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.
  1. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})$.
- Third type: Fix characteristic foliation $\mathcal{F}$, deform contact structure $\mathcal{D}$. This is used to obtain **extremal Sasaki metrics**.

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
- First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  1. Generically after this type of deformation the transverse holonomy becomes irreducible.
- Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.
  1. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})$.
- Third type: Fix characteristic foliation $\mathcal{F}$, deform contact structure $\mathcal{D}$. This is used to obtain **extremal Sasaki metrics**.
  1. This type of deformation $\eta \mapsto \eta' = \eta + d^c\zeta$, where ζ is basic and $d^c = i(\bar{\partial} - \partial)$, does not change the **isotopy class** of contact structure.

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
- First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  1. Generically after this type of deformation the transverse holonomy becomes irreducible.
- Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.
  1. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})$.
- Third type: Fix characteristic foliation $\mathcal{F}$, deform contact structure $\mathcal{D}$. This is used to obtain **extremal Sasaki metrics**.
  1. This type of deformation $\eta \mapsto \eta' = \eta + d^c\zeta$, where ζ is basic and $d^c = i(\bar{\partial} - \partial)$, does not change the **isotopy class** of contact structure.
  2. This leads to **basic cohomology** class $[d\eta]_B \in H^2_B(\mathcal{F}_\xi)$.

Transverse Geometry and Transverse Holomorphic Transformations

- Let L_ξ be the **real line bundle** generated by ξ and consider the **short exact sequence** of vector bundles $0 \longrightarrow L_\xi \longrightarrow TM \longrightarrow \nu(\mathcal{F}_\xi) \longrightarrow 0$. A choice of **Sasakian structure** $\mathcal{S} \in \mathfrak{F}(\xi)$ splits this sequence as $TM = L_\xi \oplus \mathcal{D}$ with $\mathcal{D} = \ker \eta$.

Transverse Geometry and Transverse Holomorphic Transformations

- Let L_ξ be the **real line bundle** generated by ξ and consider the **short exact sequence** of vector bundles $0 \longrightarrow L_\xi \longrightarrow TM \longrightarrow \nu(\mathcal{F}_\xi) \longrightarrow 0$. A choice of **Sasakian structure** $\mathcal{S} \in \mathfrak{F}(\xi)$ splits this sequence as $TM = L_\xi \oplus \mathcal{D}$ with $\mathcal{D} = \ker \eta$.
- The **transverse geometry** is described on the quotient bundle $\nu(\mathcal{F}_\xi)$. The endomorphism Φ induces a **complex structure** $\bar{J}$ on $\nu(\mathcal{F}_\xi)$.

Transverse Geometry and Transverse Holomorphic Transformations

- Let L_ξ be the **real line bundle** generated by ξ and consider the **short exact sequence** of vector bundles $0 \longrightarrow L_\xi \longrightarrow TM \longrightarrow \nu(\mathcal{F}_\xi) \longrightarrow 0$. A choice of **Sasakian structure** $\mathcal{S} \in \mathfrak{F}(\xi)$ splits this sequence as $TM = L_\xi \oplus \mathcal{D}$ with $\mathcal{D} = \ker \eta$.
- The **transverse geometry** is described on the quotient bundle $\nu(\mathcal{F}_\xi)$. The endomorphism Φ induces a **complex structure** $\bar{J}$ on $\nu(\mathcal{F}_\xi)$.
- If $\mathcal{S} \in \mathfrak{F}(\xi)$ the **volume** $\mathbf{V}_\mathcal{S} = \mathbf{V}_\xi = \int_M \eta \wedge (d\eta)^n$ and the **total scalar curvature** $\mathbf{S}_\mathcal{S} = \mathbf{S}_\xi = \int_M s_g\, \eta \wedge (d\eta)^n$ depend only on the class $[d\eta]_B$.

Transverse Geometry and Transverse Holomorphic Transformations

- Let L_ξ be the **real line bundle** generated by ξ and consider the **short exact sequence** of vector bundles $0\longrightarrow L_\xi \longrightarrow TM \longrightarrow \nu(\mathcal{F}_\xi)\longrightarrow 0$. A choice of **Sasakian structure** $\mathcal{S}\in\mathfrak{F}(\xi)$ splits this sequence as $TM = L_\xi\oplus\mathcal{D}$ with $\mathcal{D}=\ker\eta$.
- The **transverse geometry** is described on the quotient bundle $\nu(\mathcal{F}_\xi)$. The endomorphism Φ induces a **complex structure** $\bar{J}$ on $\nu(\mathcal{F}_\xi)$.
- If $\mathcal{S}\in\mathfrak{F}(\xi)$ the **volume** $\mathbf{V}_{\mathcal{S}}=\mathbf{V}_\xi=\int_M \eta\wedge(d\eta)^n$ and the **total scalar curvature** $\mathbf{S}_{\mathcal{S}}=\mathbf{S}_\xi=\int_M s_g\,\eta\wedge(d\eta)^n$ depend only on the class $[d\eta]_B$.
- Let $\mathfrak{Fol}(M,\mathcal{F}_\xi)$ be the subgroup of $\mathfrak{Diff}(M)$ that leaves the characteristic foliation $\mathcal{F}_\xi$ invariant. An element $\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)$ induces a map $\bar{\phi}_*:\nu(\mathcal{F}_\xi)\longrightarrow\nu(\mathcal{F}_\xi)$.

Transverse Geometry and Transverse Holomorphic Transformations

- Let L_ξ be the **real line bundle** generated by ξ and consider the **short exact sequence** of vector bundles $0\longrightarrow L_\xi\longrightarrow TM\longrightarrow\nu(\mathcal{F}_\xi)\longrightarrow 0$. A choice of **Sasakian structure** $\mathcal{S}\in\mathfrak{F}(\xi)$ splits this sequence as $TM=L_\xi\oplus\mathcal{D}$ with $\mathcal{D}=\ker\eta$.
- The **transverse geometry** is described on the quotient bundle $\nu(\mathcal{F}_\xi)$. The endomorphism Φ induces a **complex structure** $\bar{J}$ on $\nu(\mathcal{F}_\xi)$.
- If $\mathcal{S}\in\mathfrak{F}(\xi)$ the **volume** $\mathbf{V}_\mathcal{S}=\mathbf{V}_\xi=\int_M\eta\wedge(d\eta)^n$ and the **total scalar curvature** $\mathbf{S}_\mathcal{S}=\mathbf{S}_\xi=\int_M s_g\,\eta\wedge(d\eta)^n$ depend only on the class $[d\eta]_B$.
- Let $\mathfrak{Fol}(M,\mathcal{F}_\xi)$ be the subgroup of $\mathfrak{Diff}(M)$ that leaves the characteristic foliation $\mathcal{F}_\xi$ invariant. An element $\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)$ induces a map $\bar{\phi}_*:\nu(\mathcal{F}_\xi)\longrightarrow\nu(\mathcal{F}_\xi)$.
- Define the **group of transverse holomorphic transformations** by

$$\mathfrak{H}^T(\xi,\bar{J})=\{\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)\mid\bar{\phi}_*\circ\bar{J}=\bar{J}\circ\bar{\phi}_*\}.$$

Transverse Geometry and Transverse Holomorphic Transformations

- Let L_ξ be the **real line bundle** generated by ξ and consider the **short exact sequence** of vector bundles $0\longrightarrow L_\xi\longrightarrow TM\longrightarrow \nu(\mathcal{F}_\xi)\longrightarrow 0$. A choice of **Sasakian structure** $\mathcal{S}\in\mathfrak{F}(\xi)$ splits this sequence as $TM=L_\xi\oplus\mathcal{D}$ with $\mathcal{D}=\ker\eta$.
- The **transverse geometry** is described on the quotient bundle $\nu(\mathcal{F}_\xi)$. The endomorphism Φ induces a **complex structure** $\bar{J}$ on $\nu(\mathcal{F}_\xi)$.
- If $\mathcal{S}\in\mathfrak{F}(\xi)$ the **volume** $\mathbf{V}_\mathcal{S}=\mathbf{V}_\xi=\int_M\eta\wedge(d\eta)^n$ and the **total scalar curvature** $\mathbf{S}_\mathcal{S}=\mathbf{S}_\xi=\int_M s_g\,\eta\wedge(d\eta)^n$ depend only on the class $[d\eta]_B$.
- Let $\mathfrak{Fol}(M,\mathcal{F}_\xi)$ be the subgroup of $\mathfrak{Diff}(M)$ that leaves the characteristic foliation $\mathcal{F}_\xi$ invariant. An element $\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)$ induces a map $\bar{\phi}_*:\nu(\mathcal{F}_\xi)\longrightarrow\nu(\mathcal{F}_\xi)$.
- Define the **group of transverse holomorphic transformations** by

$$\mathfrak{H}^T(\xi,\bar{J})=\{\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)\mid\bar{\phi}_*\circ\bar{J}=\bar{J}\circ\bar{\phi}_*\}.$$

- The group $\mathfrak{H}^T(\xi,\bar{J})$ is **infinite dimensional**, but the quotient $\mathfrak{H}^T_0(\xi,\bar{J})=\mathfrak{H}^T(\xi,\bar{J})/\Gamma(L_\xi)$ is a **finite dimensional Lie group**.

Transverse Geometry and Transverse Holomorphic Transformations

- Let L_ξ be the **real line bundle** generated by ξ and consider the **short exact sequence** of vector bundles $0 \longrightarrow L_\xi \longrightarrow TM \longrightarrow \nu(\mathcal{F}_\xi) \longrightarrow 0$. A choice of **Sasakian structure** $\mathcal{S} \in \mathfrak{F}(\xi)$ splits this sequence as $TM = L_\xi \oplus \mathcal{D}$ with $\mathcal{D} = \ker \eta$.
- The **transverse geometry** is described on the quotient bundle $\nu(\mathcal{F}_\xi)$. The endomorphism Φ induces a **complex structure** $\bar{J}$ on $\nu(\mathcal{F}_\xi)$.
- If $\mathcal{S} \in \mathfrak{F}(\xi)$ the **volume** $\mathbf{V}_\mathcal{S} = \mathbf{V}_\xi = \int_M \eta \wedge (d\eta)^n$ and the **total scalar curvature** $\mathbf{S}_\mathcal{S} = \mathbf{S}_\xi = \int_M s_g \, \eta \wedge (d\eta)^n$ depend only on the class $[d\eta]_B$.
- Let $\mathfrak{Fol}(M, \mathcal{F}_\xi)$ be the subgroup of $\mathfrak{Diff}(M)$ that leaves the characteristic foliation $\mathcal{F}_\xi$ invariant. An element $\phi \in \mathfrak{Fol}(M, \mathcal{F}_\xi)$ induces a map $\bar{\phi}_* : \nu(\mathcal{F}_\xi) \longrightarrow \nu(\mathcal{F}_\xi)$.
- Define the **group of transverse holomorphic transformations** by

$$\mathfrak{H}^T(\xi, \bar{J}) = \{\phi \in \mathfrak{Fol}(M, \mathcal{F}_\xi) \mid \bar{\phi}_* \circ \bar{J} = \bar{J} \circ \bar{\phi}_*\}.$$

- The group $\mathfrak{H}^T(\xi, \bar{J})$ is **infinite dimensional**, but the quotient $\mathfrak{H}_0^T(\xi, \bar{J}) = \mathfrak{H}^T(\xi, \bar{J})/\Gamma(L_\xi)$ is a **finite dimensional Lie group**.
- The group $\mathfrak{H}_0^T(\xi, \bar{J})$ **does not** preserve the **contact structure** $\mathcal{D}$, but it does preserve its isotopy class.

Transverse Geometry and Transverse Holomorphic Transformations

- Let L_ξ be the **real line bundle** generated by ξ and consider the **short exact sequence** of vector bundles $0\longrightarrow L_\xi\longrightarrow TM\longrightarrow\nu(\mathcal{F}_\xi)\longrightarrow 0$. A choice of **Sasakian structure** $\mathcal{S}\in\mathfrak{F}(\xi)$ splits this sequence as $TM=L_\xi\oplus\mathcal{D}$ with $\mathcal{D}=\ker\eta$.
- The **transverse geometry** is described on the quotient bundle $\nu(\mathcal{F}_\xi)$. The endomorphism Φ induces a **complex structure** $\bar{J}$ on $\nu(\mathcal{F}_\xi)$.
- If $\mathcal{S}\in\mathfrak{F}(\xi)$ the **volume** $\mathbf{V}_\mathcal{S}=\mathbf{V}_\xi=\int_M\eta\wedge(d\eta)^n$ and the **total scalar curvature** $\mathbf{S}_\mathcal{S}=\mathbf{S}_\xi=\int_M s_g\,\eta\wedge(d\eta)^n$ depend only on the class $[d\eta]_B$.
- Let $\mathfrak{Fol}(M,\mathcal{F}_\xi)$ be the subgroup of $\mathfrak{Diff}(M)$ that leaves the characteristic foliation $\mathcal{F}_\xi$ invariant. An element $\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)$ induces a map $\bar{\phi}_*:\nu(\mathcal{F}_\xi)\longrightarrow\nu(\mathcal{F}_\xi)$.
- Define the **group of transverse holomorphic transformations** by

$$\mathfrak{H}^T(\xi,\bar{J})=\{\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)\mid\bar{\phi}_*\circ\bar{J}=\bar{J}\circ\bar{\phi}_*\}.$$

- The group $\mathfrak{H}^T(\xi,\bar{J})$ is **infinite dimensional**, but the quotient $\mathfrak{H}^T_0(\xi,\bar{J})=\mathfrak{H}^T(\xi,\bar{J})/\Gamma(L_\xi)$ is a **finite dimensional Lie group**.
- The group $\mathfrak{H}^T_0(\xi,\bar{J})$ **does not** preserve the **contact structure** $\mathcal{D}$, but it does preserve its isotopy class.
- We denote by $\mathfrak{h}^T_0(\xi,\bar{J})$ the Lie algebra of $\mathfrak{H}^T_0(\xi,\bar{J})$.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff \partial_g^\# s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^\#$ denotes the $(1,0)$ gradient with respect to g.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff$ $\partial_g^\# s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^\#$ denotes the $(1,0)$ gradient with respect to g.
- g is **extremal Sasaki metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **extremal Kähler metric**.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff$ $\partial_g^\# s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^\#$ denotes the $(1,0)$ gradient with respect to g.
- g is **extremal Sasaki metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **extremal Kähler metric**.
- g is **constant scalar curvature Sasaki metric (cscS)** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **constant scalar curvature Kähler metric (cscK)**.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff \partial_g^{\#} s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^{\#}$ denotes the $(1, 0)$ gradient with respect to g.
- g is **extremal Sasaki metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **extremal Kähler metric**.
- g is **constant scalar curvature Sasaki metric (cscS)** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **constant scalar curvature Kähler metric (cscK)**.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **extremal (or cscS)** then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff$ $\partial_g^\# s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^\#$ denotes the $(1,0)$ gradient with respect to g.
- g is **extremal Sasaki metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **extremal Kähler metric**.
- g is **constant scalar curvature Sasaki metric (cscS)** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **constant scalar curvature Kähler metric (cscK)**.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **extremal (or cscS)** then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.
- If $c_1(\mathcal{D}) = 0$ a (cscS) metric $\Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get **Sasaki-Einstein (SE)**.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff$ $\partial_g^\# s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^\#$ denotes the $(1,0)$ gradient with respect to g.
- g is **extremal Sasaki metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **extremal Kähler metric**.
- g is **constant scalar curvature Sasaki metric (cscS)** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **constant scalar curvature Kähler metric (cscK)**.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **extremal (or cscS)** then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.
- If $c_1(\mathcal{D}) = 0$ a (cscS) metric $\Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get **Sasaki-Einstein (SE)**.
- **SE metrics** have a fixed scale and g is **SE metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **Kähler-Einstein (KE) metric** with scalar curvature equal to $4n(n+1)$.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff$ $\partial_g^{\#} s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^{\#}$ denotes the $(1,0)$ gradient with respect to g.
- g is **extremal Sasaki metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **extremal Kähler metric**.
- g is **constant scalar curvature Sasaki metric (cscS)** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **constant scalar curvature Kähler metric (cscK)**.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **extremal (or cscS)** then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.
- If $c_1(\mathcal{D}) = 0$ a (cscS) metric $\Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get **Sasaki-Einstein (SE)**.
- **SE metrics** have a fixed scale and g is **SE metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **Kähler-Einstein (KE) metric** with scalar curvature equal to $4n(n+1)$.
- The **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi(X) = \int_M X(\psi_g) d\mu_g$ where $X \in \mathfrak{h}_0^T(\xi, \bar{J})$ and ψ_g is the **Ricci potential** satisfying $\rho^T = \rho_h^T + i\partial\bar{\partial}\psi_g$ where ρ^T is the **transverse Ricci form** and ρ_h^T is its **harmonic part**.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff \partial_g^\# s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^\#$ denotes the $(1,0)$ gradient with respect to g.
- g is **extremal Sasaki metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **extremal Kähler metric**.
- g is **constant scalar curvature Sasaki metric (cscS)** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **constant scalar curvature Kähler metric (cscK)**.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **extremal (or cscS)** then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.
- If $c_1(\mathcal{D}) = 0$ a (cscS) metric $\Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta, \ a, b$ constants. If $b = 0$ get **Sasaki-Einstein (SE)**.
- **SE metrics** have a fixed scale and g is **SE metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **Kähler-Einstein (KE) metric** with scalar curvature equal to $4n(n+1)$.
- The **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi(X) = \int_M X(\psi_g) d\mu_g$ where $X \in \mathfrak{h}_0^T(\xi, \bar{J})$ and ψ_g is the **Ricci potential** satisfying $\rho^T = \rho_h^T + i\partial\bar{\partial}\psi_g$ where ρ^T is the **transverse Ricci form** and ρ_h^T is its **harmonic part**.
- An extremal Sasaki metric g has **constant scalar curvature** if and only if $\mathfrak{S}\mathfrak{F} = 0$.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki **Energy functional** $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- g is a critical point of $E(g)$, called **extremal Sasaki metric**, $\iff \partial_g^\# s_g \in \mathfrak{h}_0^T(\xi, \bar{J})$, i.e. it is a transversely holomorphic vector field. Here $\partial_g^\#$ denotes the $(1,0)$ gradient with respect to g.
- g is **extremal Sasaki metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **extremal Kähler metric**.
- g is **constant scalar curvature Sasaki metric (cscS)** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **constant scalar curvature Kähler metric (cscK)**.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **extremal (or cscS)** then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.
- If $c_1(\mathcal{D}) = 0$ a (cscS) metric $\Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get **Sasaki-Einstein (SE)**.
- **SE metrics** have a fixed scale and g is **SE metric** $\iff$ the transverse metric $g_{\mathcal{D}}$ is **Kähler-Einstein (KE) metric** with scalar curvature equal to $4n(n+1)$.
- The **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi(X) = \int_M X(\psi_g) d\mu_g$ where $X \in \mathfrak{h}_0^T(\xi, \bar{J})$ and ψ_g is the **Ricci potential** satisfying $\rho^T = \rho_h^T + i\partial\bar{\partial}\psi_g$ where ρ^T is the **transverse Ricci form** and ρ_h^T is its **harmonic part**.
- An extremal Sasaki metric g has **constant scalar curvature** if and only if $\mathfrak{S}\mathfrak{F} = 0$.

Theorem

An extremal Sasaki metric g has **constant scalar curvature** *if and only if $\mathfrak{S}\mathfrak{F} = 0$.*

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)
- **SE** metrics on **irregular** Sasakian manifolds (Gauntlett, Martelli, Sparks, Waldram (2004))

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)
- **SE** metrics on **irregular** Sasakian manifolds (Gauntlett, Martelli, Sparks, Waldram (2004))
- **SE** metrics on **spheres**, including **exotic spheres** B-,Galicki, Kollár (Thomas) (2005)

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)
- **SE** metrics on **irregular** Sasakian manifolds (Gauntlett, Martelli, Sparks, Waldram (2004))
- **SE** metrics on **spheres**, including **exotic spheres** B-,Galicki, Kollár (Thomas) (2005)
- **SE** on many 5-manifolds (Kollár; B-,Galicki, Ghigi-Kollár (2005-07))

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)
- **SE** metrics on **irregular** Sasakian manifolds (Gauntlett, Martelli, Sparks, Waldram (2004))
- **SE** metrics on **spheres**, including **exotic spheres** B-,Galicki, Kollár (Thomas) (2005)
- **SE** on many 5-manifolds (Kollár; B-,Galicki, Ghigi-Kollár (2005-07))
- Existence of **SE** metrics in Sasaki cone of **toric contact manifolds of Reeb type** with $c_1 = 0$ by deforming in Sasaki cone (Futaki,Ono,Wang (2009)); Uniqueness (Cho,Futaki,Ono (2008))

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)
- **SE** metrics on **irregular** Sasakian manifolds (Gauntlett, Martelli, Sparks, Waldram (2004))
- **SE** metrics on **spheres**, including **exotic spheres** B-,Galicki, Kollár (Thomas) (2005)
- **SE** on many 5-manifolds (Kollár; B-,Galicki, Ghigi-Kollár (2005-07))
- Existence of **SE** metrics in Sasaki cone of **toric contact manifolds of Reeb type** with $c_1 = 0$ by deforming in Sasaki cone (Futaki,Ono,Wang (2009)); Uniqueness (Cho,Futaki,Ono (2008))
- Uniqueness of **SE** metrics in the transverse Kähler class up to transverse holomorphic transformations (Nitta,Sekiya (2012))

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)
- **SE** metrics on **irregular** Sasakian manifolds (Gauntlett, Martelli, Sparks, Waldram (2004))
- **SE** metrics on **spheres**, including **exotic spheres** B-,Galicki, Kollár (Thomas) (2005)
- **SE** on many 5-manifolds (Kollár; B-,Galicki, Ghigi-Kollár (2005-07))
- Existence of **SE** metrics in Sasaki cone of **toric contact manifolds of Reeb type** with $c_1 = 0$ by deforming in Sasaki cone (Futaki,Ono,Wang (2009)); Uniqueness (Cho,Futaki,Ono (2008))
- Uniqueness of **SE** metrics in the transverse Kähler class up to transverse holomorphic transformations (Nitta,Sekiya (2012))
- New examples of **SE** metrics by deforming in the Sasaki cone (Mabuchi,Nakagawa (2013))

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)
- **SE** metrics on **irregular** Sasakian manifolds (Gauntlett, Martelli, Sparks, Waldram (2004))
- **SE** metrics on **spheres**, including **exotic spheres** B-,Galicki, Kollár (Thomas) (2005)
- **SE** on many 5-manifolds (Kollár; B-,Galicki, Ghigi-Kollár (2005-07))
- Existence of **SE** metrics in Sasaki cone of **toric contact manifolds of Reeb type** with $c_1 = 0$ by deforming in Sasaki cone (Futaki,Ono,Wang (2009)); Uniqueness (Cho,Futaki,Ono (2008))
- Uniqueness of **SE** metrics in the transverse Kähler class up to transverse holomorphic transformations (Nitta,Sekiya (2012))
- New examples of **SE** metrics by deforming in the Sasaki cone (Mabuchi,Nakagawa (2013))
- Geometry and topology of **SE** metrics by deforming in Sasaki cone (B-,Tønnesen-Friedman (2015))

The Sasaki-Einstein Problem: Some History

- The general **extremal problem** involves a **4th order PDE**, whereas, the easier **SE problem** involves the **Monge-Ampère equation**, a second order PDE.
- **3-Sasakian manifolds** are automatically **Sasaki-Einstein (SE)** (Kashiwada (1971))
- **3-Sasakian manifolds** over quaternionic Kähler manifolds (Ishihara, Konishi (1972-75))
- Many **3-Sasakian** examples B-,Galicki, Mann (Rees) (1994,1998)
- **SE** metrics on S^1 orbibundles over **Kähler-Einstein (KE)** orbifolds. B-,Galicki (2000)
- **SE** metrics on certain connected sums of $S^2 \times S^3$ B-,Galicki, Nakamaye (2002-3)
- **SE** metrics on **irregular** Sasakian manifolds (Gauntlett, Martelli, Sparks, Waldram (2004))
- **SE** metrics on **spheres**, including **exotic spheres** B-,Galicki, Kollár (Thomas) (2005)
- **SE** on many 5-manifolds (Kollár; B-,Galicki, Ghigi-Kollár (2005-07))
- Existence of **SE** metrics in Sasaki cone of **toric contact manifolds of Reeb type** with $c_1 = 0$ by deforming in Sasaki cone (Futaki,Ono,Wang (2009)); Uniqueness (Cho,Futaki,Ono (2008))
- Uniqueness of **SE** metrics in the transverse Kähler class up to transverse holomorphic transformations (Nitta,Sekiya (2012))
- New examples of **SE** metrics by deforming in the Sasaki cone (Mabuchi,Nakagawa (2013))
- Geometry and topology of **SE** metrics by deforming in Sasaki cone (B-,Tønnesen-Friedman (2015))
- The Donaldson,Tian, Yau conjecture in the SE case is still open.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1, l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- Take $M_1 = M$ a regular Sasaki manifold, $M_2 = S^3_{\mathbf{w}}$, weighted 3-sphere with $\mathbf{w} = (w_1, w_2)$.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- Take $M_1 = M$ a regular Sasaki manifold, $M_2 = S^3_{\mathbf{w}}$, weighted 3-sphere with $\mathbf{w} = (w_1, w_2)$.
- Our main object of study is the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ which is a smooth manifold if $\gcd(l_2, l_1 w_i) = 1$.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- Take $M_1 = M$ a regular Sasaki manifold, $M_2 = S^3_{\mathbf{w}}$, weighted 3-sphere with $\mathbf{w} = (w_1, w_2)$.
- Our main object of study is the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ which is a smooth manifold if $\gcd(l_2, l_1 w_i) = 1$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Hodge manifold.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- Take $M_1 = M$ a regular Sasaki manifold, $M_2 = S^3_{\mathbf{w}}$, weighted 3-sphere with $\mathbf{w} = (w_1, w_2)$.
- Our main object of study is the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ which is a smooth manifold if $\gcd(l_2, l_1 w_i) = 1$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Hodge manifold.
- Take N with a constant scalar curvature metric. $\mathbb{CP}^1[\mathbf{w}]$ has an extremal Kähler metric $\Rightarrow$ $N \times \mathbb{CP}^1[\mathbf{w}] \Rightarrow M \star_{l_1,l_2} S^3_{\mathbf{w}}$ has an **extremal Sasaki metric**.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- Take $M_1 = M$ a regular Sasaki manifold, $M_2 = S^3_{\mathbf{w}}$, weighted 3-sphere with $\mathbf{w} = (w_1, w_2)$.
- Our main object of study is the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ which is a smooth manifold if $\gcd(l_2, l_1 w_i) = 1$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Hodge manifold.
- Take N with a constant scalar curvature metric. $\mathbb{CP}^1[\mathbf{w}]$ has an extremal Kähler metric $\Rightarrow$ $N \times \mathbb{CP}^1[\mathbf{w}] \Rightarrow M \star_{l_1,l_2} S^3_{\mathbf{w}}$ has an **extremal Sasaki metric**.
- There is a 2-dimensional subcone $\mathfrak{t}^+_{\mathbf{w}}$ of the Sasaki cone $\mathfrak{t}^+$ of $M \star_{l_1,l_2} S^3_{\mathbf{w}}$.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- Take $M_1 = M$ a regular Sasaki manifold, $M_2 = S^3_{\mathbf{w}}$, weighted 3-sphere with $\mathbf{w} = (w_1, w_2)$.
- Our main object of study is the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ which is a smooth manifold if $\gcd(l_2, l_1 w_i) = 1$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Hodge manifold.
- Take N with a constant scalar curvature metric. $\mathbb{CP}^1[\mathbf{w}]$ has an extremal Kähler metric $\Rightarrow$ $N \times \mathbb{CP}^1[\mathbf{w}] \Rightarrow M \star_{l_1,l_2} S^3_{\mathbf{w}}$ has an **extremal Sasaki metric**.
- There is a 2-dimensional subcone $\mathfrak{t}^+_{\mathbf{w}}$ of the Sasaki cone $\mathfrak{t}^+$ of $M \star_{l_1,l_2} S^3_{\mathbf{w}}$.
- The join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ can be realized as a **lens space bundle** over N, fiber $= L(l_2; l_1 w_1, l_1 w_2)$.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- Take $M_1 = M$ a regular Sasaki manifold, $M_2 = S^3_{\mathbf{w}}$, weighted 3-sphere with $\mathbf{w} = (w_1, w_2)$.
- Our main object of study is the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ which is a smooth manifold if $\gcd(l_2, l_1 w_i) = 1$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Hodge manifold.
- Take N with a constant scalar curvature metric. $\mathbb{CP}^1[\mathbf{w}]$ has an extremal Kähler metric $\Rightarrow$ $N \times \mathbb{CP}^1[\mathbf{w}] \Rightarrow M \star_{l_1,l_2} S^3_{\mathbf{w}}$ has an **extremal Sasaki metric**.
- There is a 2-dimensional subcone $\mathfrak{t}^+_{\mathbf{w}}$ of the Sasaki cone $\mathfrak{t}^+$ of $M \star_{l_1,l_2} S^3_{\mathbf{w}}$.
- The join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ can be realized as a **lens space bundle** over N, fiber $= L(l_2; l_1 w_1, l_1 w_2)$.
- Given the differentials in the **spectral sequence** of the fibration $M \longrightarrow N \longrightarrow \mathsf{B}S^1$ get **cohomology ring** of $M_1 \star_{l_1,l_2} S^3_{\mathbf{w}}$ using **commutative diagram** of fibrations:

$$
\begin{array}{ccccc}
M \times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathsf{B}S^1 \\
\Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\psi} \\
M \times S^3_{\mathbf{w}} & \longrightarrow & N \times \mathsf{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathsf{B}S^1 \times \mathsf{B}S^1
\end{array}
$$

where $\mathsf{B}G$ is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.

The $S^3_{\mathbf{w}}$-Join Construction (B-,Tønnesen-Friedman

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- Take $M_1 = M$ a regular Sasaki manifold, $M_2 = S^3_{\mathbf{w}}$, weighted 3-sphere with $\mathbf{w} = (w_1, w_2)$.
- Our main object of study is the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ which is a smooth manifold if $\gcd(l_2, l_1 w_i) = 1$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Hodge manifold.
- Take N with a constant scalar curvature metric. $\mathbb{CP}^1[\mathbf{w}]$ has an extremal Kähler metric $\Rightarrow$ $N \times \mathbb{CP}^1[\mathbf{w}] \Rightarrow M \star_{l_1,l_2} S^3_{\mathbf{w}}$ has an **extremal Sasaki metric**.
- There is a 2-dimensional subcone $\mathfrak{t}^+_{\mathbf{w}}$ of the Sasaki cone $\mathfrak{t}^+$ of $M \star_{l_1,l_2} S^3_{\mathbf{w}}$.
- The join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ can be realized as a **lens space bundle** over N, fiber $= L(l_2; l_1 w_1, l_1 w_2)$.
- Given the differentials in the **spectral sequence** of the fibration $M \longrightarrow N \longrightarrow \mathrm{B}S^1$ get **cohomology ring** of $M_1 \star_{l_1,l_2} S^3_{\mathbf{w}}$ using **commutative diagram** of fibrations:

$$\begin{array}{ccccc} M \times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathrm{B}S^1 \\ \downarrow{=} & & \downarrow & & \downarrow{\psi} \\ M \times S^3_{\mathbf{w}} & \longrightarrow & N \times \mathrm{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathrm{B}S^1 \times \mathrm{B}S^1 \end{array}$$

  where $\mathrm{B}G$ is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.
- Example: $M = S^{2p+1}$ get $H^*(M_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) \approx \mathbb{Z}[x, y]/(w_1 w_2 l_1^2 x^2, x^{p+1}, x^2 y, y^2)$ with $\deg x = 2$, $\deg y = 2p + 1$. l_1, w_1, w_2 fixed $\Rightarrow$ (Sullivan) **finite number of diffeomorphism types**.

The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions (B-,Tønnesen-Friedman

- The existence of an extra **Hamiltonian Killing** vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki **w**-cone $\mathfrak{t}^+_{\mathbf{w}}$.

The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions (B-,Tønnesen-Friedman

- The existence of an extra **Hamiltonian Killing** vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki **w**-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.

The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions (B-,Tønnesen-Friedman

- The existence of an extra **Hamiltonian Killing** vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1, mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1, mv_2})$

The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions (B-,Tønnesen-Friedman

- The existence of an extra **Hamiltonian Killing** vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(1\!\!1 \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.

- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.

The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions (B-,Tønnesen-Friedman

- The existence of an extra **Hamiltonian Killing** vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

  consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})^{d_N}$ is a $(d_N + 3)$ order ($(d_N + 2)$ order) **polynomial** we get **extremal (CSC) Kähler metrics**. Here d_N is the complex dimension of N.

The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions (B-,Tønnesen-Friedman

- The existence of an extra **Hamiltonian Killing** vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})^{d_N}$ is a $(d_N + 3)$ order ($(d_N + 2)$ order) **polynomial** we get **extremal (CSC) Kähler metrics**. Here d_N is the complex dimension of N.
- Lifing to $M_{l_1,l_2,\mathbf{w}}$ gives **extremal (CSC) Sasaki metrics** in the quasi-regular case.

The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions (B-,Tønnesen-Friedman

- The existence of an extra **Hamiltonian Killing** vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.

- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})^{d_N}$ is a $(d_N + 3)$ order ($(d_N + 2)$ order) **polynomial** we get **extremal (CSC) Kähler metrics**. Here d_N is the complex dimension of N.
- Lifing to $M_{l_1,l_2,\mathbf{w}}$ gives **extremal (CSC) Sasaki metrics** in the quasi-regular case.
- The **irregular** case uses a continuity argument together with the fact that quasi-regular Sasaki structures are dense in the Sasaki cone.

The $S^3_{\mathbf{w}}$-Join and Admissible Kähler Constructions (B-,Tønnesen-Friedman

- The existence of an extra **Hamiltonian Killing** vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

  consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})^{d_N}$ is a $(d_N + 3)$ order ($(d_N + 2)$ order) **polynomial** we get **extremal (CSC) Kähler metrics**. Here d_N is the complex dimension of N.
- Lifing to $M_{l_1,l_2,\mathbf{w}}$ gives **extremal (CSC) Sasaki metrics** in the quasi-regular case.
- The **irregular** case uses a continuity argument together with the fact that quasi-regular Sasaki structures are dense in the Sasaki cone.
- The $\mathbf{w}$-cone can have several **CSC** Sasaki rays which comes from a sign changing count.

Einstein-Hilbert Functional (B-,Huang,Legendre,Tønnesen-Friedman)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.

Einstein-Hilbert Functional (B-,Huang,Legendre,Tønnesen-Friedman)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- The **total scalar curvature** as a functional restricted to the Sasaki cone: $\mathbf{S}_\xi = \int_M s_g$

Einstein-Hilbert Functional (B-,Huang,Legendre,Tønnesen-Friedman)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- The **total scalar curvature** as a functional restricted to the Sasaki cone: $\mathbf{S}_\xi = \int_M s_g$
- Want invariance under **transverse homothety** so we define the **Einstein-Hilbert Functional**

$$\mathbf{H}(\xi) = \frac{\mathbf{S}_\xi^{n+1}}{\mathbf{V}_\xi^{n}}$$

where $\mathbf{V}$ is the total volume.

Einstein-Hilbert Functional (B-,Huang,Legendre,Tønnesen-Friedman)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- The **total scalar curvature** as a functional restricted to the Sasaki cone: $\mathbf{S}_\xi = \int_M s_g$
- Want invariance under **transverse homothety** so we define the **Einstein-Hilbert Functional**

$$\mathbf{H}(\xi) = \frac{\mathbf{S}_\xi^{n+1}}{\mathbf{V}_\xi^n}$$

where $\mathbf{V}$ is the total volume.

- Variation gives

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n} \mathfrak{S}\mathfrak{F}_\xi \circ \Phi.$$

Einstein-Hilbert Functional (B-,Huang,Legendre,Tønnesen-Friedman)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- The **total scalar curvature** as a functional restricted to the Sasaki cone: $\mathbf{S}_\xi = \int_M s_g$
- Want invariance under **transverse homothety** so we define the **Einstein-Hilbert Functional**

$$\mathbf{H}(\xi) = \frac{\mathbf{S}_\xi^{n+1}}{\mathbf{V}_\xi^n}$$

where $\mathbf{V}$ is the total volume.

- Variation gives

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n} \mathfrak{S}\mathfrak{F}_\xi \circ \Phi.$$

- So **critical points** of $\mathbf{H}(\xi)$ are the zeros of the total scalar curvature $\mathbf{S}_\xi$ and the zeros of the **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi$.

Einstein-Hilbert Functional (B-,Huang,Legendre,Tønnesen-Friedman)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- The **total scalar curvature** as a functional restricted to the Sasaki cone: $\mathbf{S}_\xi = \int_M s_g$
- Want invariance under **transverse homothety** so we define the **Einstein-Hilbert Functional**

$$\mathbf{H}(\xi) = \frac{\mathbf{S}_\xi^{n+1}}{\mathbf{V}_\xi^n}$$

where $\mathbf{V}$ is the total volume.

- Variation gives

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n} \mathfrak{S}\mathfrak{F}_\xi \circ \Phi.$$

- So **critical points** of $\mathbf{H}(\xi)$ are the zeros of the total scalar curvature $\mathbf{S}_\xi$ and the zeros of the **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi$.
- This gives **constant scalar curvature Sasaki metrics (cscS)** are critical points of $\mathbf{H}(\xi)$.

Einstein-Hilbert Functional (B-,Huang,Legendre,Tønnesen-Friedman)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- The **total scalar curvature** as a functional restricted to the Sasaki cone: $\mathbf{S}_\xi = \int_M s_g$
- Want invariance under **transverse homothety** so we define the **Einstein-Hilbert Functional**

$$\mathbf{H}(\xi) = \frac{\mathbf{S}_\xi^{n+1}}{\mathbf{V}_\xi^n}$$

where $\mathbf{V}$ is the total volume.

- Variation gives

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n} \mathfrak{S}\mathfrak{F}_\xi \circ \Phi.$$

- So **critical points** of $\mathbf{H}(\xi)$ are the zeros of the total scalar curvature $\mathbf{S}_\xi$ and the zeros of the **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi$.
- This gives **constant scalar curvature Sasaki metrics (cscS)** are critical points of $\mathbf{H}(\xi)$.
- Sasaki version of the Donaldson-Tian-Yau **conjecture**: **cscS** $\iff$ cone is **K-polystable**.

Sasakian K-Stability

- The notions of **K-stability, K-polystability, K-semistability** have been amply developed in Kählerian geometry. Here we only deal with **K-semistability**. A Sasaki manifold $(M,\mathcal{S})$ is said to be **K-semistable** if the affine cone $(C(M),I)$ is **K-semistable** (Collins,Székelyhidi).

Sasakian K-Stability

- The notions of **K-stability, K-polystability, K-semistability** have been amply developed in Kählerian geometry. Here we only deal with **K-semistability**. A Sasaki manifold $(M, \mathcal{S})$ is said to be **K-semistable** if the affine cone $(C(M), I)$ is **K-semistable** (Collins,Székelyhidi).
- To define **K-semistability** we need the notion of **test configuration**. Let $Y = C(M)$ be the affine cone of a Sasaki manifold M and suppose it has an action of a complex torus $T_{\mathbb{C}}$ such that its Lie algebra $\mathfrak{t}_{\mathbb{C}}$ contains the Reeb vector field ξ. We view the pair (Y, ξ) as a **polarized variety**. A $T_{\mathbb{C}}$**-equivariant test configuration** for Y is given by a set of k $T_{\mathbb{C}}$-homogeneous generators $f_1, \ldots, f_k$ of the coordinate ring of Y and k integers $w_1, \ldots, w_k$. The functions $f_1, \ldots, f_k$ are generators of $H^0(Y, \mathcal{O}_Y)$.

Sasakian K-Stability

- The notions of **K-stability, K-polystability, K-semistability** have been amply developed in Kählerian geometry. Here we only deal with **K-semistability**. A Sasaki manifold $(M, \mathcal{S})$ is said to be **K-semistable** if the affine cone $(C(M), I)$ is **K-semistable** (Collins,Székelyhidi).
- To define **K-semistability** we need the notion of **test configuration**. Let $Y = C(M)$ be the affine cone of a Sasaki manifold M and suppose it has an action of a complex torus $T_{\mathbb{C}}$ such that its Lie algebra $\mathfrak{t}_{\mathbb{C}}$ contains the Reeb vector field ξ. We view the pair (Y, ξ) as a **polarized variety**. A $T_{\mathbb{C}}$**-equivariant test configuration** for Y is given by a set of k $T_{\mathbb{C}}$-homogeneous generators $f_1, \ldots, f_k$ of the coordinate ring of Y and k integers $w_1, \ldots, w_k$. The functions $f_1, \ldots, f_k$ are generators of $H^0(Y, \mathcal{O}_Y)$.
- Using the functions $f_1, \ldots, f_k$ embed Y into $\mathbb{C}^k$ and consider the $\mathbb{C}^*$ action on $\mathbb{C}^k$ with weights $w_1, \ldots, w_k$.

Sasakian K-Stability

- The notions of **K-stability, K-polystability, K-semistability** have been amply developed in Kählerian geometry. Here we only deal with **K-semistability**. A Sasaki manifold $(M, \mathcal{S})$ is said to be **K-semistable** if the affine cone $(C(M), I)$ is **K-semistable** (Collins,Székelyhidi).
- To define **K-semistability** we need the notion of **test configuration**. Let $Y = C(M)$ be the affine cone of a Sasaki manifold M and suppose it has an action of a complex torus $T_{\mathbb{C}}$ such that its Lie algebra $\mathfrak{t}_{\mathbb{C}}$ contains the Reeb vector field ξ. We view the pair (Y, ξ) as a **polarized variety**. A $T_{\mathbb{C}}$**-equivariant test configuration** for Y is given by a set of k $T_{\mathbb{C}}$-homogeneous generators $f_1, \ldots, f_k$ of the coordinate ring of Y and k integers $w_1, \ldots, w_k$. The functions $f_1, \ldots, f_k$ are generators of $H^0(Y, \mathcal{O}_Y)$.
- Using the functions $f_1, \ldots, f_k$ embed Y into $\mathbb{C}^k$ and consider the $\mathbb{C}^*$ action on $\mathbb{C}^k$ with weights $w_1, \ldots, w_k$.
- Taking the flat limit Y_0 over $\mathbb{C}$ of the $\mathbb{C}^*$ orbits gives an **affine scheme** $\mathcal{Y}$ over $\mathbb{C}$. We then get an action of $\mathbb{C}^*$ on the 'central fiber' Y_0.

- The notions of **K-stability, K-polystability, K-semistability** have been amply developed in Kählerian geometry. Here we only deal with **K-semistability**. A Sasaki manifold $(M, \mathcal{S})$ is said to be **K-semistable** if the affine cone $(C(M), I)$ is **K-semistable** (Collins,Székelyhidi).
- To define **K-semistability** we need the notion of **test configuration**. Let $Y = C(M)$ be the affine cone of a Sasaki manifold M and suppose it has an action of a complex torus $T_{\mathbb{C}}$ such that its Lie algebra $\mathfrak{t}_{\mathbb{C}}$ contains the Reeb vector field ξ. We view the pair (Y, ξ) as a **polarized variety**. A $T_{\mathbb{C}}$**-equivariant test configuration** for Y is given by a set of k $T_{\mathbb{C}}$-homogeneous generators $f_1, \ldots, f_k$ of the coordinate ring of Y and k integers $w_1, \ldots, w_k$. The functions $f_1, \ldots, f_k$ are generators of $H^0(Y, \mathcal{O}_Y)$.
- Using the functions $f_1, \ldots, f_k$ embed Y into $\mathbb{C}^k$ and consider the $\mathbb{C}^*$ action on $\mathbb{C}^k$ with weights $w_1, \ldots, w_k$.
- Taking the flat limit Y_0 over $\mathbb{C}$ of the $\mathbb{C}^*$ orbits gives an **affine scheme** $\mathcal{Y}$ over $\mathbb{C}$. We then get an action of $\mathbb{C}^*$ on the 'central fiber' Y_0.
- The **Donaldson-Futaki invariant** of such a test configuration is defined by

$$Fut(Y_0, \xi, a) := \frac{\mathbf{V}_\xi}{n} D_a \left(\frac{\mathbf{S}_\xi}{\mathbf{V}_\xi} \right) + \frac{\mathbf{S}_\xi D_a \mathbf{V}_\xi}{n(n+1)\mathbf{V}_\xi}$$

where D_a is the **directional derivative**.

- The notions of **K-stability, K-polystability, K-semistability** have been amply developed in Kählerian geometry. Here we only deal with **K-semistability**. A Sasaki manifold $(M, \mathcal{S})$ is said to be **K-semistable** if the affine cone $(C(M), I)$ is **K-semistable** (Collins,Székelyhidi).
- To define **K-semistability** we need the notion of **test configuration**. Let $Y = C(M)$ be the affine cone of a Sasaki manifold M and suppose it has an action of a complex torus $T_{\mathbb{C}}$ such that its Lie algebra $\mathfrak{t}_{\mathbb{C}}$ contains the Reeb vector field ξ. We view the pair (Y, ξ) as a **polarized variety**. A $T_{\mathbb{C}}$**-equivariant test configuration** for Y is given by a set of k $T_{\mathbb{C}}$-homogeneous generators $f_1, \ldots, f_k$ of the coordinate ring of Y and k integers $w_1, \ldots, w_k$. The functions $f_1, \ldots, f_k$ are generators of $H^0(Y, \mathcal{O}_Y)$.
- Using the functions $f_1, \ldots, f_k$ embed Y into $\mathbb{C}^k$ and consider the $\mathbb{C}^*$ action on $\mathbb{C}^k$ with weights $w_1, \ldots, w_k$.
- Taking the flat limit Y_0 over $\mathbb{C}$ of the $\mathbb{C}^*$ orbits gives an **affine scheme** $\mathcal{Y}$ over $\mathbb{C}$. We then get an action of $\mathbb{C}^*$ on the 'central fiber' Y_0.
- The **Donaldson-Futaki invariant** of such a test configuration is defined by

$$Fut(Y_0, \xi, a) := \frac{\mathbf{V}_\xi}{n} D_a\left(\frac{\mathbf{S}_\xi}{\mathbf{V}_\xi}\right) + \frac{\mathbf{S}_\xi D_a \mathbf{V}_\xi}{n(n+1)\mathbf{V}_\xi}$$

  where D_a is the **directional derivative**.
- We say that (Y, ξ) is **K-semistable** if for each $T_{\mathbb{C}}$ with $\xi \in \mathfrak{t}$ and any $T_{\mathbb{C}}$**-equivariant test configuration** $Fut(Y_0, \xi, a) \geq 0$. It is **K-polystable** if equality holds only if it is a product configuration, that is, Y_0 is isomorphic to Y.

The Einstein-Hilbert Functional and K-stability [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (1)

The set of **critical points** *of the Einstein–Hilbert functional* $\mathbf{H}_\xi$ *is the union of the* **zeros** *of the* **Sasaki-Futaki invariant** *and of the* **total transversal scalar curvature**. *In particular, if a Reeb vector field admits a* **compatible cscS metric** *then it is a* **critical point of the Einstein–Hilbert functional**.

- Proof: This follows immediately from

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\mathfrak{S}\mathfrak{F}_\xi \circ \Phi. \tag{1}$$

□

The Einstein-Hilbert Functional and K-stability [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (1)

The set of **critical points** *of the Einstein–Hilbert functional* $\mathbf{H}_\xi$ *is the union of the* **zeros** *of the* **Sasaki-Futaki invariant** *and of the* **total transversal scalar curvature**. *In particular, if a Reeb vector field admits a* **compatible cscS metric** *then it is a* **critical point of the Einstein–Hilbert functional**.

- Proof: This follows immediately from

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\mathfrak{S}\mathfrak{F}_\xi \circ \Phi. \tag{1}$$

□

Theorem (2)

If ξ *is not a critical point of* $\mathbf{H}(\xi)$ *then* (Y,ξ) *is not K-semistable. Alternatively, if* (Y,ξ) *is* **K-semistable** *then* ξ *is a critical point of the Einstein-Hilbert functional* $\mathbf{H}(\xi)$, *and if in addition* $\mathbf{S}_\xi^n \neq 0$, *then the Sasaki–Futaki invariant* $\mathfrak{S}\mathfrak{F}_\xi$ *vanishes identically.*

The Einstein-Hilbert Functional and K-stability [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (1)

The set of **critical points** *of the Einstein–Hilbert functional* $\mathbf{H}_\xi$ *is the union of the* **zeros** *of the* **Sasaki-Futaki invariant** *and of the* **total transversal scalar curvature***. In particular, if a Reeb vector field admits a* **compatible cscS metric** *then it is a* **critical point of the Einstein–Hilbert functional***.*

- Proof: This follows immediately from

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\mathfrak{S}\mathfrak{F}_\xi \circ \Phi. \tag{1}$$

□

Theorem (2)

If ξ *is not a critical point of* $\mathbf{H}(\xi)$ *then* (Y,ξ) *is not K-semistable. Alternatively, if* (Y,ξ) *is* **K-semistable** *then* ξ *is a critical point of the Einstein-Hilbert functional* $\mathbf{H}(\xi)$*, and if in addition* $\mathbf{S}_\xi^n \neq 0$*, then the Sasaki–Futaki invariant* $\mathfrak{S}\mathfrak{F}_\xi$ *vanishes identically.*

- Outline of Proof: The linearity in a in the equation for the Donaldson-Futaki invariant says that K-semistability implies $Fut(Y_0,\xi,a) = 0$ for any product test configuration.

The Einstein-Hilbert Functional and K-stability [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (1)

The set of **critical points** *of the Einstein–Hilbert functional* $\mathbf{H}_\xi$ *is the union of the* **zeros** *of the* **Sasaki-Futaki invariant** *and of the* **total transversal scalar curvature**. *In particular, if a Reeb vector field admits a* **compatible cscS metric** *then it is a* **critical point of the Einstein–Hilbert functional**.

- Proof: This follows immediately from

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\mathfrak{S}\mathfrak{F}_\xi \circ \Phi. \tag{1}$$

□

Theorem (2)

If ξ *is not a critical point of* $\mathbf{H}(\xi)$ *then* (Y,ξ) *is not K-semistable. Alternatively, if* (Y,ξ) *is* **K-semistable** *then* ξ *is a critical point of the Einstein-Hilbert functional* $\mathbf{H}(\xi)$*, and if in addition* $\mathbf{S}_\xi^n \neq 0$*, then the Sasaki–Futaki invariant* $\mathfrak{S}\mathfrak{F}_\xi$ *vanishes identically.*

- Outline of Proof: The linearity in a in the equation for the Donaldson-Futaki invariant says that K-semistability implies $Fut(Y_0,\xi,a)=0$ for any product test configuration.
- Calculation gives $Fut(Y_0,\xi,a)=c_n\mathfrak{S}\mathfrak{F}_\xi(\Phi(a))$ (Tipler, van Coevering).

The Einstein-Hilbert Functional and K-stability [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (1)

The set of **critical points** *of the Einstein–Hilbert functional* $\mathbf{H}_\xi$ *is the union of the* **zeros** *of the* **Sasaki-Futaki invariant** *and of the* **total transversal scalar curvature**. *In particular, if a Reeb vector field admits a* **compatible cscS metric** *then it is a* **critical point of the Einstein–Hilbert functional**.

- Proof: This follows immediately from

$$d\mathbf{H}_\xi = \frac{n(n+1)\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\mathfrak{S}\mathfrak{F}_\xi \circ \Phi. \quad (1)$$

□

Theorem (2)

If ξ *is not a critical point of* $\mathbf{H}(\xi)$ *then* (Y,ξ) *is not K-semistable. Alternatively, if* (Y,ξ) *is* **K-semistable** *then* ξ *is a critical point of the Einstein-Hilbert functional* $\mathbf{H}(\xi)$*, and if in addition* $\mathbf{S}_\xi^n \neq 0$*, then the Sasaki–Futaki invariant* $\mathfrak{S}\mathfrak{F}_\xi$ *vanishes identically.*

- Outline of Proof: The linearity in a in the equation for the Donaldson-Futaki invariant says that K-semistability implies $Fut(Y_0,\xi,a)=0$ for any product test configuration.
- Calculation gives $Fut(Y_0,\xi,a) = c_n\mathfrak{S}\mathfrak{F}_\xi(\Phi(a))$ (Tipler, van Coevering).
- So if ξ is not a critical point of $\mathbf{H}(\xi)$, then there exists $a\in\mathfrak{t}$ such that $Fut(Y_0,\xi,a)\neq 0$ for any test configuration. The theorem then follows by contradiction. □

Stability Theorems continued [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (3)

Suppose the **Sasaki cone** *is* **exhausted by extremal Sasaki metrics** *and that the total transverse scalar curvature does not vanish. Then the set of critical points of the* **Einstein-Hilbert functional** *is precisely the set of rays in the Sasaki cone with* **constant scalar curvature***. In particular, in this case a Sasakian structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *has* **constant scalar curvature** *if and only if* (Y, ξ) *is* **K-semistable***.*

Stability Theorems continued [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (3)

Suppose the **Sasaki cone** *is* **exhausted by extremal Sasaki metrics** *and that the total transverse scalar curvature does not vanish. Then the set of critical points of the* **Einstein-Hilbert functional** *is precisely the set of rays in the Sasaki cone with* **constant scalar curvature**. *In particular, in this case a Sasakian structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *has* **constant scalar curvature** *if and only if* (Y, ξ) *is* **K-semistable**.

- Proof: This follows from Theorem 2, work of Collins, Székelyhidi, and the vanishing of the Sasaki-Futaki invariant for **cscS metrics**. □

Stability Theorems continued [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (3)

Suppose the **Sasaki cone** *is* **exhausted by extremal Sasaki metrics** *and that the total transverse scalar curvature does not vanish. Then the set of critical points of the* **Einstein-Hilbert functional** *is precisely the set of rays in the Sasaki cone with* **constant scalar curvature***. In particular, in this case a Sasakian structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *has* **constant scalar curvature** *if and only if* (Y, ξ) *is* **K-semistable***.*

- Proof: This follows from Theorem 2, work of Collins, Székelyhidi, and the vanishing of the Sasaki-Futaki invariant for **cscS metrics**. □

Theorem (4)

Let M *be a regular Sasaki manifold with constant transverse scalar curvature, and consider the* $S^3_{\mathbf{w}}$*-join* $M_{l_1,l_2,\mathbf{w}}$*. Its* **w***-cone* $\mathfrak{t}^+_{\mathbf{w}}$ *has a cscS ray* $\mathfrak{r}_\xi$ *if and only if the* **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi$ *vanishes on the Lie algebra* $\mathfrak{t}_{\mathbf{w}} \otimes \mathbb{C}$*. Then for* $\xi \in \mathfrak{t}^+_{\mathbf{w}}$ *the polarized affine cone* (Y, ξ) *associated to* $M_{l_1,l_2,\mathbf{w}}$ *is* **K-semistable** *if and only if the Sasakian structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *on* $M_{l_1,l_2,\mathbf{w}}$ *has* **constant scalar curvature***. There can be several* **cscS** *rays in* $\mathfrak{t}^+_{\mathbf{w}}$*.*

Theorem (3)

Suppose the **Sasaki cone** *is* **exhausted by extremal Sasaki metrics** *and that the total transverse scalar curvature does not vanish. Then the set of critical points of the* **Einstein-Hilbert functional** *is precisely the set of rays in the Sasaki cone with* **constant scalar curvature**. *In particular, in this case a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ has* **constant scalar curvature** *if and only if (Y, ξ) is* **K-semistable**.

- Proof: This follows from Theorem 2, work of Collins, Székelyhidi, and the vanishing of the Sasaki-Futaki invariant for **cscS metrics**. □

Theorem (4)

Let M be a regular Sasaki manifold with constant transverse scalar curvature, and consider the $S^3_{\mathbf{w}}$-join $M_{l_1,l_2,\mathbf{w}}$. Its $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$ has a cscS ray $\mathfrak{r}_\xi$ if and only if the **Sasaki-Futaki invariant** *$\mathfrak{S}\mathfrak{F}_\xi$ vanishes on the Lie algebra $\mathfrak{t}_{\mathbf{w}} \otimes \mathbb{C}$. Then for $\xi \in \mathfrak{t}^+_{\mathbf{w}}$ the polarized affine cone (Y, ξ) associated to $M_{l_1,l_2,\mathbf{w}}$ is* **K-semistable** *if and only if the Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M_{l_1,l_2,\mathbf{w}}$ has* **constant scalar curvature**. *There can be several* **cscS** *rays in $\mathfrak{t}^+_{\mathbf{w}}$.*

- Outline of Proof: First some notation: H is a Hamiltonian vector field on $M_{l_1,l_2,\mathbf{w}}$, $\xi = \xi_{\mathbf{v}}$ with $\mathbf{v} = (v_1, v_2) \in \mathfrak{t}^+_{\mathbf{w}}$, and $b = v_1/v_2$.

Theorem (3)

Suppose the **Sasaki cone** *is* **exhausted by extremal Sasaki metrics** *and that the total transverse scalar curvature does not vanish. Then the set of critical points of the* **Einstein-Hilbert functional** *is precisely the set of rays in the Sasaki cone with* **constant scalar curvature**. *In particular, in this case a Sasakian structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *has* **constant scalar curvature** *if and only if* (Y, ξ) *is* **K-semistable**.

- Proof: This follows from Theorem 2, work of Collins, Székelyhidi, and the vanishing of the Sasaki-Futaki invariant for **cscS metrics**. □

Theorem (4)

Let M *be a regular Sasaki manifold with constant transverse scalar curvature, and consider the* $S^3_{\mathbf{w}}$*-join* $M_{l_1,l_2,\mathbf{w}}$. *Its* **w***-cone* $\mathfrak{t}^+_{\mathbf{w}}$ *has a cscS ray* $\mathfrak{r}_\xi$ *if and only if the* **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi$ *vanishes on the Lie algebra* $\mathfrak{t}_{\mathbf{w}} \otimes \mathbb{C}$. *Then for* $\xi \in \mathfrak{t}^+_{\mathbf{w}}$ *the polarized affine cone* (Y, ξ) *associated to* $M_{l_1,l_2,\mathbf{w}}$ *is* **K-semistable** *if and only if the Sasakian structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *on* $M_{l_1,l_2,\mathbf{w}}$ *has* **constant scalar curvature**. *There can be several* **cscS** *rays in* $\mathfrak{t}^+_{\mathbf{w}}$.

- Outline of Proof: First some notation: H is a Hamiltonian vector field on $M_{l_1,l_2,\mathbf{w}}$, $\xi = \xi_{\mathbf{v}}$ with $\mathbf{v} = (v_1, v_2) \in \mathfrak{t}^+_{\mathbf{w}}$, and $b = v_1/v_2$.
- Generally the **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi$ is difficult to compute, but

Stability Theorems continued [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (3)

Suppose the **Sasaki cone** *is* **exhausted by extremal Sasaki metrics** *and that the total transverse scalar curvature does not vanish. Then the set of critical points of the* **Einstein-Hilbert functional** *is precisely the set of rays in the Sasaki cone with* **constant scalar curvature**. *In particular, in this case a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ has* **constant scalar curvature** *if and only if (Y, ξ) is* **K-semistable**.

- Proof: This follows from Theorem 2, work of Collins, Székelyhidi, and the vanishing of the Sasaki-Futaki invariant for **cscS metrics**. □

Theorem (4)

Let M be a regular Sasaki manifold with constant transverse scalar curvature, and consider the $S^3_{\mathbf{w}}$-join $M_{l_1,l_2,\mathbf{w}}$. Its $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$ has a cscS ray $\mathfrak{r}_\xi$ if and only if the **Sasaki-Futaki invariant** *$\mathfrak{S}\mathfrak{F}_\xi$ vanishes on the Lie algebra $\mathfrak{t}_{\mathbf{w}} \otimes \mathbb{C}$. Then for $\xi \in \mathfrak{t}^+_{\mathbf{w}}$ the polarized affine cone (Y, ξ) associated to $M_{l_1,l_2,\mathbf{w}}$ is* **K-semistable** *if and only if the Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M_{l_1,l_2,\mathbf{w}}$ has* **constant scalar curvature**. *There can be several* **cscS** *rays in $\mathfrak{t}^+_{\mathbf{w}}$.*

- Outline of Proof: First some notation: H is a Hamiltonian vector field on $M_{l_1,l_2,\mathbf{w}}$, $\xi = \xi_{\mathbf{v}}$ with $\mathbf{v} = (v_1, v_2) \in \mathfrak{t}^+_{\mathbf{w}}$, and $b = v_1/v_2$.
- Generally the **Sasaki-Futaki invariant** $\mathfrak{S}\mathfrak{F}_\xi$ is difficult to compute, but
- On $M_{l_1,l_2,\mathbf{w}}$ the **Einstein-Hilbert functional** $\mathbf{H}(b)$ is easy to compute.

Outline of Proof of Theorem 4

- The **Einstein-Hilbert functional** takes the form

$$\mathbf{H}(b) = \frac{\left(l_1 w_1^{d_N+1} b^{d_N+2} + (l_2 A - l_1 w_2) w_1^{d_N} b^{d_N+1} + (l_1 w_1 - l_2 A) w_2^{d_N} b - l_1 w_2^{d_N+1}\right)^{d_N+2}}{(w_1 b - w_2)\left(w_1^{d_N+1} b^{d_N+2} - w_2^{d_N+1} b\right)^{d_N+1}}.$$

Outline of Proof of Theorem 4

- The **Einstein-Hilbert functional** takes the form

$$\mathbf{H}(b) = \frac{\left(l_1 w_1^{d_N+1} b^{d_N+2} + (l_2 A - l_1 w_2) w_1^{d_N} b^{d_N+1} + (l_1 w_1 - l_2 A) w_2^{d_N} b - l_1 w_2^{d_N+1}\right)^{d_N+2}}{(w_1 b - w_2)\left(w_1^{d_N+1} b^{d_N+2} - w_2^{d_N+1} b\right)^{d_N+1}}.$$

- And $\mathbf{H}'(b) = \frac{\mathbf{s}_\xi^{d_N+1}}{(bv_1)^{2d_N+3}\mathbf{V}_\xi} \frac{f(b)}{(w_1 b - w_2)^3}$ giving $\mathfrak{S}\mathfrak{F}_{\xi_{\mathbf{v}}}(\Phi H) = \frac{f(b)}{v_2^{2n+1}(w_1 b - w_2)^3 \mathbf{V}_{\xi_{\mathbf{v}}}}$ where

$$\begin{aligned}
f(b) &= (d_N+1) l_1 w_1^{2d_N+3} b^{2d_N+4} \\
&- w_1^{2(d_N+1)} b^{2d_N+3} (A l_2 + l_1 (d_N+1) w_2) \\
&+ w_1^{d_N+2} w_2^{d_N} b^{d_N+3} ((d_N+1)(A(d_N+1) l_2 - l_1((d_N+1) w_1 + (d_N+2) w_2))) \\
&- w_1^{d_N+1} w_2^{d_N+1} b^{d_N+2} (2A d_N (d_N+2) l_2 - (d_N+1)(2d_N+3) l_1 (w_1 + w_2)) \\
&+ w_1^{d_N} w_2^{d_N+2} b^{d_N+1} (d_N+1)(A(d_N+1) l_2 - l_1((d_N+2) w_1 + (d_N+1) w_2)) \\
&- w_2^{2(d_N+1)} (b(A l_2 + l_1 (d_N+1) w_1)) \\
&+ (d_N+1) l_1 w_2^{2d_N+3}.
\end{aligned}$$

□

Critical Points of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (5)

Assume that the **total transversal scalar curvature** *is sign definite and bounded away from* 0 *on each transversal set of the* **Sasaki cone***, then there exists at least one* **Reeb vector field** *for which the* **Sasaki–Futaki invariant** $\mathfrak{SF}$ *vanishes identically.*

Critical Points of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (5)

Assume that the **total transversal scalar curvature** *is sign definite and bounded away from* 0 *on each transversal set of the* **Sasaki cone**, *then there exists at least one* **Reeb vector field** *for which the* **Sasaki–Futaki invariant** $\mathfrak{SF}$ *vanishes identically.*

- Proof: Assume the total scalar curvature is sign definite and bounded away from zero on any transversal subset of $\mathfrak{t}^+$ which gives $|\mathbf{S}_\xi| \geq m_\xi \mathbf{V}_\xi, \quad \forall \xi \in \mathfrak{t}^+$ and some constant m_ξ depending only on ξ.

Critical Points of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (5)

Assume that the **total transversal scalar curvature** *is sign definite and bounded away from* 0 *on each transversal set of the* **Sasaki cone**, *then there exists at least one* **Reeb vector field** *for which the* **Sasaki–Futaki invariant** $\mathfrak{S}\mathfrak{F}$ *vanishes identically.*

- Proof: Assume the total scalar curvature is sign definite and bounded away from zero on any transversal subset of $\mathfrak{t}^+$ which gives $|\mathbf{S}_\xi| \geq m_\xi \mathbf{V}_\xi, \quad \forall \xi \in \mathfrak{t}^+$ and some constant m_ξ depending only on ξ.
- Assume first that $\mathbf{S}_\xi > 0$ the above bound gives $\mathbf{H}(\xi) \geq m_\xi^{n+1} \mathbf{V}_\xi$.

Critical Points of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (5)

Assume that the **total transversal scalar curvature** *is sign definite and bounded away from* 0 *on each transversal set of the* **Sasaki cone**, *then there exists at least one* **Reeb vector field** *for which the* **Sasaki–Futaki invariant** $\mathfrak{S}\mathfrak{F}$ *vanishes identically.*

- Proof: Assume the total scalar curvature is sign definite and bounded away from zero on any transversal subset of $\mathfrak{t}^+$ which gives $|\mathbf{S}_\xi| \geq m_\xi \mathbf{V}_\xi, \quad \forall \xi \in \mathfrak{t}^+$ and some constant m_ξ depending only on ξ.
- Assume first that $\mathbf{S}_\xi > 0$ the above bound gives $\mathbf{H}(\xi) \geq m_\xi^{n+1}\mathbf{V}_\xi$.
- But the left hand side is invariant under scaling of ξ, so there exists an m_0 independent of ξ such that on a transverse subset Σ we have $\mathbf{H}(\xi) \geq m_\xi^{n+1}\mathbf{V}_\xi \geq m_o^{m+1}\mathbf{V}_\xi \quad \forall \xi \in \Sigma$.

Critical Points of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (5)

Assume that the **total transversal scalar curvature** *is sign definite and bounded away from* 0 *on each transversal set of the* **Sasaki cone**, *then there exists at least one* **Reeb vector field** *for which the* **Sasaki–Futaki invariant** $\mathfrak{SF}$ *vanishes identically.*

- Proof: Assume the total scalar curvature is sign definite and bounded away from zero on any transversal subset of $\mathfrak{t}^+$ which gives $|\mathbf{S}_\xi| \geq m_\xi \mathbf{V}_\xi, \quad \forall \xi \in \mathfrak{t}^+$ and some constant m_ξ depending only on ξ.
- Assume first that $\mathbf{S}_\xi > 0$ the above bound gives $\mathbf{H}(\xi) \geq m_\xi^{n+1}\mathbf{V}_\xi$.
- But the left hand side is invariant under scaling of ξ, so there exists an m_0 independent of ξ such that on a transverse subset Σ we have $\mathbf{H}(\xi) \geq m_\xi^{n+1}\mathbf{V}_\xi \geq m_o^{m+1}\mathbf{V}_\xi \quad \forall \xi \in \Sigma$.
- A result of He, Sun says that $\mathbf{V}_\xi$ is strictly convex on Σ and tends to infinity on the boundary $\partial\Sigma$.

Critical Points of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (5)

Assume that the **total transversal scalar curvature** *is sign definite and bounded away from* 0 *on each transversal set of the* **Sasaki cone**, *then there exists at least one* **Reeb vector field** *for which the* **Sasaki–Futaki invariant** $\mathfrak{SF}$ *vanishes identically.*

- Proof: Assume the total scalar curvature is sign definite and bounded away from zero on any transversal subset of $\mathfrak{t}^+$ which gives $|\mathbf{S}_\xi| \geq m_\xi \mathbf{V}_\xi, \quad \forall \xi \in \mathfrak{t}^+$ and some constant m_ξ depending only on ξ.
- Assume first that $\mathbf{S}_\xi > 0$ the above bound gives $\mathbf{H}(\xi) \geq m_\xi^{n+1}\mathbf{V}_\xi$.
- But the left hand side is invariant under scaling of ξ, so there exists an m_0 independent of ξ such that on a transverse subset Σ we have $\mathbf{H}(\xi) \geq m_\xi^{n+1}\mathbf{V}_\xi \geq m_o^{m+1}\mathbf{V}_\xi \quad \forall \xi \in \Sigma$.
- A result of He, Sun says that $\mathbf{V}_\xi$ is strictly convex on Σ and tends to infinity on the boundary $\partial\Sigma$.
- So $\mathbf{H}(\xi)$ must reach a minimum somewhere in Σ and the variational formula implies $\mathfrak{SF} = 0$. □

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

- Lemma: Let (M^{2n+1}, D, J, g, ξ) be a cscS compact manifold of non-zero transversal scalar curvature, and $T \subset \mathfrak{CR}(D, J) \cap \mathrm{Isom}(g)$ be the maximal compact torus with Lie algebra $\mathfrak{t}$.

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

- Lemma: Let (M^{2n+1}, D, J, g, ξ) be a cscS compact manifold of non-zero transversal scalar curvature, and $T \subset \mathfrak{CR}(D, J) \cap \mathrm{Isom}(g)$ be the maximal compact torus with Lie algebra $\mathfrak{t}$.
  1. If the transverse scalar curvature is negative, the E-H functional **H** is convex if n is even and concave if n is odd.

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

- Lemma: Let (M^{2n+1}, D, J, g, ξ) be a cscS compact manifold of non-zero transversal scalar curvature, and $T \subset \mathfrak{CR}(D, J) \cap \mathrm{Isom}(g)$ be the maximal compact torus with Lie algebra $\mathfrak{t}$.
  1. If the transverse scalar curvature is negative, the E-H functional **H** is convex if n is even and concave if n is odd.
  2. If the transverse scalar curvature is positive and the first non-zero eigenvalue λ_1 of the Laplacian, restricted to the space of T–invariant functions is bounded below by $\frac{s_g^T}{2n+1}$, then **H** is convex near ξ.

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

- Lemma: Let (M^{2n+1}, D, J, g, ξ) be a cscS compact manifold of non-zero transversal scalar curvature, and $T \subset \mathfrak{CR}(D, J) \cap \mathrm{Isom}(g)$ be the maximal compact torus with Lie algebra $\mathfrak{t}$.
  1. If the transverse scalar curvature is negative, the E-H functional **H** is convex if n is even and concave if n is odd.
  2. If the transverse scalar curvature is positive and the first non-zero eigenvalue λ_1 of the Laplacian, restricted to the space of T–invariant functions is bounded below by $\frac{s_g^T}{2n+1}$, then **H** is convex near ξ.
- Moreover, in both cases the **cscS ray** is isolated in the Sasaki cone.

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

- Lemma: Let (M^{2n+1}, D, J, g, ξ) be a cscS compact manifold of non-zero transversal scalar curvature, and $T \subset \mathfrak{CR}(D, J) \cap \mathrm{Isom}(g)$ be the maximal compact torus with Lie algebra $\mathfrak{t}$.
  1. If the transverse scalar curvature is negative, the E-H functional **H** is convex if n is even and concave if n is odd.
  2. If the transverse scalar curvature is positive and the first non-zero eigenvalue λ_1 of the Laplacian, restricted to the space of T–invariant functions is bounded below by $\frac{s_g^T}{2n+1}$, then **H** is convex near ξ.
- Moreover, in both cases the **cscS ray** is isolated in the Sasaki cone.
- The lemma follows from the second variational formula at a cscS metric:

$$\frac{d^2}{dt^2}_{t=0}\mathbf{H}(\xi_t) = n(n+1)(2n+1)(s_g^T)^n \left(\|d(\eta(a))\|_g^2 - \frac{s_g^T}{(2n+1)}\|\eta(a)\|_g^2\right)$$

and the well known inequality $\|d(\eta(a))\|_g^2 \geq \lambda_1\|\eta(a)\|_g^2$. □

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

- Lemma: Let (M^{2n+1}, D, J, g, ξ) be a cscS compact manifold of non-zero transversal scalar curvature, and $T \subset \mathfrak{CR}(D, J) \cap \mathrm{Isom}(g)$ be the maximal compact torus with Lie algebra $\mathfrak{t}$.
  1. If the transverse scalar curvature is negative, the E-H functional **H** is convex if n is even and concave if n is odd.
  2. If the transverse scalar curvature is positive and the first non-zero eigenvalue λ_1 of the Laplacian, restricted to the space of T–invariant functions is bounded below by $\frac{s_g^T}{2n+1}$, then **H** is convex near ξ.
- Moreover, in both cases the **cscS ray** is isolated in the Sasaki cone.
- The lemma follows from the second variational formula at a cscS metric:

$$\frac{d^2}{dt^2}_{t=0}\mathbf{H}(\xi_t) = n(n+1)(2n+1)(s_g^T)^n \left(\|d(\eta(a))\|_g^2 - \frac{s_g^T}{(2n+1)}\|\eta(a)\|_g^2 \right)$$

  and the well known inequality $\|d(\eta(a))\|_g^2 \geq \lambda_1 \|\eta(a)\|_g^2$. □
- Proof of Theorem (6): For negative transverse scalar curvature, the theorem follows immediately from (1) of the lemma.

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

- Lemma: Let (M^{2n+1}, D, J, g, ξ) be a cscS compact manifold of non-zero transversal scalar curvature, and $T \subset \mathfrak{CR}(D, J) \cap \mathrm{Isom}(g)$ be the maximal compact torus with Lie algebra $\mathfrak{t}$.
  1. If the transverse scalar curvature is negative, the E-H functional **H** is convex if n is even and concave if n is odd.
  2. If the transverse scalar curvature is positive and the first non-zero eigenvalue λ_1 of the Laplacian, restricted to the space of T–invariant functions is bounded below by $\frac{s_g^T}{2n+1}$, then **H** is convex near ξ.
- Moreover, in both cases the **cscS ray** is isolated in the Sasaki cone.
- The lemma follows from the second variational formula at a cscS metric:

$$\frac{d^2}{dt^2}_{t=0}\mathbf{H}(\xi_t) = n(n+1)(2n+1)(s_g^T)^n\left(\|d(\eta(a))\|_g^2 - \frac{s_g^T}{(2n+1)}\|\eta(a)\|_g^2\right)$$

  and the well known inequality $\|d(\eta(a))\|_g^2 \geq \lambda_1\|\eta(a)\|_g^2$. □
- Proof of Theorem (6): For negative transverse scalar curvature, the theorem follows immediately from (1) of the lemma.
- For the **Sasaki-η–Einstein** case a bound $\lambda_1 \geq \frac{s_g^T}{2n}$ due to Matsushima together with (2) of the lemma gives the result. □

Convexity of E-H Functional [B-,Huang,Legendre,Tønnesen-Friedman]

Theorem (6)

Let (M, D, J, g, ξ) be either a **cscS** *compact manifold of* **negative transverse scalar curvature** *or a compact* **Sasaki-η–Einstein** *manifold of positive transverse scalar curvature. Then* **H** *is* **transversally convex** *at ξ (or* **transversally concave** *in the negative cscS case with n odd).*

- Lemma: Let (M^{2n+1}, D, J, g, ξ) be a cscS compact manifold of non-zero transversal scalar curvature, and $T \subset \mathfrak{CR}(D, J) \cap \mathrm{Isom}(g)$ be the maximal compact torus with Lie algebra $\mathfrak{t}$.
  1. If the transverse scalar curvature is negative, the E-H functional **H** is convex if n is even and concave if n is odd.
  2. If the transverse scalar curvature is positive and the first non-zero eigenvalue λ_1 of the Laplacian, restricted to the space of T–invariant functions is bounded below by $\frac{s_g^T}{2n+1}$, then **H** is convex near ξ.
- Moreover, in both cases the **cscS ray** is isolated in the Sasaki cone.
- The lemma follows from the second variational formula at a cscS metric:

$$\frac{d^2}{dt^2}_{t=0}\mathbf{H}(\xi_t) = n(n+1)(2n+1)(s_g^T)^n\left(\|d(\eta(a))\|_g^2 - \frac{s_g^T}{(2n+1)}\|\eta(a)\|_g^2\right)$$

  and the well known inequality $\|d(\eta(a))\|_g^2 \geq \lambda_1\|\eta(a)\|_g^2$. □
- Proof of Theorem (6): For negative transverse scalar curvature, the theorem follows immediately from (1) of the lemma.
- For the **Sasaki-η–Einstein** case a bound $\lambda_1 \geq \frac{s_g^T}{2n}$ due to Matsushima together with (2) of the lemma gives the result. □
- This lemma leads to the general question: Are the rays of **constant scalar curvature Sasaki metrics** isolated in the Sasaki cone?

THANK YOU FOR YOUR ATTENTION!!