The Kähler Geometry of Bott Manifolds

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August 25, 2017
Gauge Theories, Monopoles, Moduli Spaces and Integrable Systems
A Conference honouring Jacques Hurtubise on his 60th birthday
Montreal, Quebec, Canada

BON ANNIVERSAIRE JACQUES

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5. They are best approached through the notion of a **Bott Tower** which we now describe.

Bott Towers

1. Consider **Closed Complex Manifolds** M_k for $k = 0, 1, \ldots, n$ with $M_0 = \{pt\}$ and M_k the total space of the $\mathbb{CP}^1$-bundle $\pi_k \colon \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_k) \to M_{k-1}$ giving the sequence

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5. The **Quotient Construction**: Any **Bott tower** is obtained from the **complex torus action** $(t_i)_{i=1}^n \in (\mathbb{C}^*)^n$ on $(z_j, w_j)_{j=1}^n \in (\mathbb{C}^2 \setminus \{0\})^n$ by

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6. The **Cohomology Ring**: $H^*(M_n, \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \dots, x_n]/\mathcal{I}$ where $\mathcal{I}$ is generated by $x_k y_k$ with $y_k = \sum_{j=1}^n A_k^j x_j$.

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7. **Problem**: When does the **cohomology ring** determine the **diffeomorphism type**? (Choi, Masuda, Panov, Suh)

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8. $\mathcal{K}$ is isomorphic to the **ample cone** $\mathcal{A}$ of $\mathbb{T}^n$ **invariant ample divisors**.

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- Delzant's Theorem $\Rightarrow$ (M^{2n}, ω, J) is a smooth projective toric variety.

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- **INGREDIENTS OF PROOF**:
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- There is a **bijection** between the set of **Bott manifolds** compatible with (M^{2n}, ω) and the set of $\mathcal{G}_1^{BT}$ **orbits** in $\mathcal{G}_0^{BT}$ with an element compatible with ω.
- Elements of $\mathcal{G}_0^{BT}/\mathcal{G}_1^{BT}$ have **distinct complex structures**.
- Then a **cohomological rigidity** result of Choi-Suh and Masuda-Panov gives a **bijection** between the set of **Bott manifolds compatible** with (M^{2n}, ω) and the set of T^n invariant integrable **complex structures** J compatible with (M^{2n}, ω).
- Delzant's Theorem $\Rightarrow$ (M^{2n}, ω, J) is a smooth projective toric variety.
- The corresponding **Delzant polytope** P is **combinatorially equivalent** to n cube.

Symplectic Structures

- Given a **Bott tower** $M_n(A)$ choose a $\mathbb{T}^n$ invariant compatible symplectic form ω. Then say that the **symplectic manifold** (M^{2n}, ω) is of **Bott type**
- $N_B(M^{2n}, \omega)$ denotes the **number** of $\mathbb{T}^n$ invariant **complex structures** that are compatible with (M^{2n}, ω) which is isomorphic to the **number** of compatible **Bott manifolds**.
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- **smooth projective toric varieties** $\{M_{\mathcal{F}}\} \approx \{\mathcal{F}_M\}$ **smooth normal fans** $\mathcal{F}$ over $\{P\}$. $\square$

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- Given a **principal** $\mathbb{T}^\ell$ bundle $\mathfrak{G}\longrightarrow\mathbb{CP}^1$ construct the associated fiber bundle $M = \mathfrak{G} \times_{\mathbb{T}^\ell} V$ with **fiber** V where V is a compact **toric Kähler manifold** of complex dimension ℓ.

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Lemma (Apostolov, Calderbank, Gauduchon, Tønnesen-Friedmann)

If V admits an extremal **toric Kähler metric** g_V*, then M admits* **compatible extremal Kähler** *metrics (at least in some Kähler classes).*

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Theorem (3)

Let $M_n(A)$ *be a* **Bott tower**. *If the elements below the diagonal of any row of the lower triangular unipotent matrix* A *all have the same sign, then* $M_n(A)$ **does not admit** *a Kähler metric with* **constant scalar curvature**. *In particular, if* $A_2^1 \neq 0$ *then* $M_n(A)$ **does not admit** *a Kähler metric with* **constant scalar curvature**.

- The proof essentially follows from Demazure's Theorem by computing possible root vectors.

The Twist of Bott Manifolds

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- The only 0 twist **Fano Bott manifold** is the product $(\mathbb{CP}^1)^n$.

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- Much of this case recovers previous work of Koiso,Sakane,Guan,Hwang, and Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman

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- There is an **uncountable** number of **extremal almost Kähler metrics** (non integrable) on **stage 3 Bott manifolds**. These are not **constant scalar curvature**

Thank You

THANK YOU FOR YOUR ATTENTION

and

HAPPY BIRTHDAY JACQUES