Extremal Sasakian Geometry:
the Join and Admissible Constructions

Charles Boyer

University of New Mexico

September 2014

Fundamental Problems

Talk based on joint work with Christina Tønnesen-Friedman

Problems: (1) Given a manifold determine how many contact structures of Sasaki type there are.

(2) Given a contact structure or isotopy class of contact structures:

- Determine the space of compatible Sasakian structures.

Fundamental Problems

Talk based on joint work with Christina Tønnesen-Friedman

Problems: (1) Given a manifold determine how many contact structures of Sasaki type there are.

(2) Given a contact structure or isotopy class of contact structures:

- Determine the space of compatible Sasakian structures.
- Determine the (pre)-moduli space of extremal Sasakian structures.

Fundamental Problems

Talk based on joint work with Christina Tønnesen-Friedman

Problems: (1) Given a manifold determine how many contact structures of Sasaki type there are.

(2) Given a contact structure or isotopy class of contact structures:

- Determine the space of compatible Sasakian structures.
- Determine the (pre)-moduli space of extremal Sasakian structures.
- Determine those of constant scalar curvature (cscS). How many?

Fundamental Problems

Talk based on joint work with Christina Tønnesen-Friedman

Problems: (1) Given a manifold determine how many contact structures of Sasaki type there are.

(2) Given a contact structure or isotopy class of contact structures:

- Determine the space of compatible Sasakian structures.
- Determine the (pre)-moduli space of extremal Sasakian structures.
- Determine those of constant scalar curvature (cscS). How many?
- Determine the (pre)-moduli space of Sasakian structures with the same underlying CR structure.

Fundamental Problems

Talk based on joint work with Christina Tønnesen-Friedman

Problems: (1) Given a manifold determine how many contact structures of Sasaki type there are.

(2) Given a contact structure or isotopy class of contact structures:

- Determine the space of compatible Sasakian structures.
- Determine the (pre)-moduli space of extremal Sasakian structures.
- Determine those of constant scalar curvature (cscS). How many?
- Determine the (pre)-moduli space of Sasakian structures with the same underlying CR structure.
- Determine those having distinct underlying CR structures.

Fundamental Problems

Talk based on joint work with Christina Tønnesen-Friedman

Problems: (1) Given a manifold determine how many contact structures of Sasaki type there are.

(2) Given a contact structure or isotopy class of contact structures:

- Determine the space of compatible Sasakian structures.
- Determine the (pre)-moduli space of extremal Sasakian structures.
- Determine those of constant scalar curvature (cscS). How many?
- Determine the (pre)-moduli space of Sasakian structures with the same underlying CR structure.
- Determine those having distinct underlying CR structures.

- We give partial answers to these problems for particular cases that are obtained by combining the Sasaki join construction of B-, Galicki, Ornea with the admissible Kähler construction of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman.

Contact manifold

• **Compact Contact Manifold** M.

- A **contact 1-form** η such that

$$\eta \wedge (d\eta)^n \neq 0.$$

Contact manifold

• **Compact Contact Manifold** M.

- A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
- defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure.

Contact manifold

• Compact Contact Manifold M.

- A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
- defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure.
- A **contact invariant**: the first Chern class $c_1(\mathcal{D})$

Contact manifold

• Compact Contact Manifold M.

- A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
- defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure.
- A **contact invariant**: the first Chern class $c_1(\mathcal{D})$
- Unique vector field ξ, called the **Reeb vector field**, satisfying
$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

Contact manifold

• Compact Contact Manifold M.

- A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
- defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure.
- A **contact invariant**: the first Chern class $c_1(\mathcal{D})$
- Unique vector field ξ, called the **Reeb vector field**, satisfying
$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$
- The **characteristic foliation** $\mathcal{F}_\xi$: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k = 1$; otherwise, it is **irregular**.

Contact manifold

• Compact Contact Manifold M.

- A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
- defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure.
- A **contact invariant**: the first Chern class $c_1(\mathcal{D})$
- Unique vector field ξ, called the **Reeb vector field**, satisfying
$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

- The **characteristic foliation** $\mathcal{F}_\xi$: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k = 1$; otherwise, it is **irregular**.
- Quasi-regularity is strong, most contact 1-forms are irregular.

Contact manifold

• **Compact Contact Manifold M.**

- A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
- defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure.
- A **contact invariant**: the first Chern class $c_1(\mathcal{D})$
- Unique vector field ξ, called the **Reeb vector field**, satisfying
$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

- The **characteristic foliation** $\mathcal{F}_\xi$: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k = 1$; otherwise, it is **irregular**.
- Quasi-regularity is strong, most contact 1-forms are irregular.
- Contact bundle $\mathcal{D} \to$ choose **almost complex structure** J extend to an endomorphism Φ with $\Phi\xi = 0$ with a compatible metric
$$g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$$
Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**

Contact manifold

• Compact Contact Manifold M.

- A **contact 1-form** η such that
$$\eta \wedge (d\eta)^n \neq 0.$$
- defines a **contact structure**
$$\eta' \sim \eta \iff \eta' = f\eta$$
for some $f \neq 0$, take $f > 0$, or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\,\eta$ of TM with a conformal symplectic structure.
- A **contact invariant**: the first Chern class $c_1(\mathcal{D})$
- Unique vector field ξ, called the **Reeb vector field**, satisfying
$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

- The **characteristic foliation** $\mathcal{F}_\xi$: It is called **quasi-regular** if each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times. It is **regular** if $k = 1$; otherwise, it is **irregular**.
- Quasi-regularity is strong, most contact 1-forms are irregular.
- Contact bundle $\mathcal{D} \to$ choose **almost complex structure** J extend to an endomorphism Φ with $\Phi\xi = 0$ with a compatible metric
$$g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$$
Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**
- The pair $(\mathcal{D}, J)$ is a **strictly pseudo-convex almost CR structure** (sψCR structure).

Sasakian Structures and Symmetries

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_\mathcal{D}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- A Sasakian structure determines a **strictly pseudo-convex CR structure** $(\mathcal{D}, J)$.

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- A Sasakian structure determines a **strictly pseudo-convex CR structure** $(\mathcal{D}, J)$.
- Nested structures: **Sasakian** $\subset$ **strictly pseudo-convex CR** $\subset$ **Contact**

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- A Sasakian structure determines a **strictly pseudo-convex CR structure** $(\mathcal{D}, J)$.
- Nested structures: **Sasakian** $\subset$ **strictly pseudo-convex CR** $\subset$ **Contact**
- Nested symmetries: $\mathfrak{Aut}(\mathcal{S}) \subset \mathfrak{CR}(\mathcal{D}, J) \subset \mathfrak{Con}(M, \mathcal{D})$

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- A Sasakian structure determines a **strictly pseudo-convex CR structure** $(\mathcal{D}, J)$.
- Nested structures: **Sasakian** $\subset$ **strictly pseudo-convex CR** $\subset$ **Contact**
- Nested symmetries: $\mathfrak{Aut}(\mathcal{S}) \subset \mathfrak{CR}(\mathcal{D}, J) \subset \mathfrak{Con}(M, \mathcal{D})$
  1. **Contactomorphism Group**: $\mathfrak{Con}(M, \mathcal{D}) = \{\phi \in \mathfrak{Diff}(M) \mid \phi_* \mathcal{D} \subset \mathcal{D}\}$

Sasakian Structures and Symmetries

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_\mathcal{D}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- A Sasakian structure determines a **strictly pseudo-convex CR structure** $(\mathcal{D}, J)$.
- Nested structures: **Sasakian** $\subset$ **strictly pseudo-convex CR** $\subset$ **Contact**
- Nested symmetries: $\mathfrak{Aut}(\mathcal{S}) \subset \mathfrak{CR}(\mathcal{D}, J) \subset \mathfrak{Con}(M, \mathcal{D})$
  1. **Contactomorphism Group**: $\mathfrak{Con}(M, \mathcal{D}) = \{\phi \in \mathfrak{Diff}(M) \mid \phi_* \mathcal{D} \subset \mathcal{D}\}$
  2. **CR automorphism group**: $\mathfrak{CR}(\mathcal{D}, J) = \{\phi \in \mathfrak{Con}(M, \mathcal{D}) \mid \phi_* J = J \phi_*\}$

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- A Sasakian structure determines a **strictly pseudo-convex CR structure** $(\mathcal{D}, J)$.
- Nested structures: **Sasakian** $\subset$ **strictly pseudo-convex CR** $\subset$ **Contact**
- Nested symmetries: $\mathfrak{Aut}(\mathcal{S}) \subset \mathfrak{CR}(\mathcal{D}, J) \subset \mathfrak{Con}(M, \mathcal{D})$
  1. **Contactomorphism Group**: $\mathfrak{Con}(M, \mathcal{D}) = \{\phi \in \mathfrak{Diff}(M) \mid \phi_* \mathcal{D} \subset \mathcal{D}\}$
  2. **CR automorphism group**: $\mathfrak{CR}(\mathcal{D}, J) = \{\phi \in \mathfrak{Con}(M, \mathcal{D}) \mid \phi_* J = J\phi_*\}$
  3. **Sasakian automorphism group**: $\mathfrak{Aut}(\mathcal{S}) = \{\phi \in \mathfrak{CR}(\mathcal{D}, J) \mid \phi_* \xi = \xi,\ \phi^* g = g\}$

Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

- A Sasakian structure determines a **strictly pseudo-convex CR structure** $(\mathcal{D}, J)$.
- Nested structures: **Sasakian** $\subset$ **strictly pseudo-convex CR** $\subset$ **Contact**
- Nested symmetries: $\mathfrak{Aut}(\mathcal{S}) \subset \mathfrak{CR}(\mathcal{D}, J) \subset \mathfrak{Con}(M, \mathcal{D})$
  1. **Contactomorphism Group**: $\mathfrak{Con}(M, \mathcal{D}) = \{\phi \in \mathfrak{Diff}(M) \mid \phi_* \mathcal{D} \subset \mathcal{D}\}$
  2. **CR automorphism group**: $\mathfrak{CR}(\mathcal{D}, J) = \{\phi \in \mathfrak{Con}(M, \mathcal{D}) \mid \phi_* J = J\phi_*\}$
  3. **Sasakian automorphism group**: $\mathfrak{Aut}(\mathcal{S}) = \{\phi \in \mathfrak{CR}(\mathcal{D}, J) \mid \phi_* \xi = \xi,\ \phi^* g = g\}$
- **maximal torus**: $T^k \subset \mathfrak{Aut}(\mathcal{S}) \subset \mathfrak{CR}(\mathcal{D}, J) \subset \mathfrak{Con}(M, \mathcal{D})$ with $1 \leq k \leq n+1$

Sasaki cones and bouquets

- **Sasaki cones**

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D},J)=\{\xi'\in\mathfrak{t}_k\mid \eta(\xi')>0,\}$ s.t. $\mathcal{S}=(\xi,\eta,\Phi,g)\in(\mathcal{D},J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D},J)=\mathfrak{t}_k^+(\mathcal{D},J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D},J)$

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D},J)=\{\xi'\in\mathfrak{t}_k \mid \eta(\xi')>0,\}$ s.t. $\mathcal{S}=(\xi,\eta,\Phi,g)\in(\mathcal{D},J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D},J)=\mathfrak{t}_k^+(\mathcal{D},J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D},J)$
  4. $\kappa(\mathcal{D},J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D},J)$
  5. $1\leq \dim\kappa(\mathcal{D},J)\leq n+1$ and if $\dim\kappa(\mathcal{D},J)=n+1$, M is **toric Sasakian**

Sasaki cones and bouquets

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n + 1$ and if $\dim \kappa(\mathcal{D}, J) = n + 1$, M is **toric Sasakian**
- **Sasaki bouquets**

Sasaki cones and bouquets

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D},J)=\{\xi'\in\mathfrak{t}_k \mid \eta(\xi')>0,\}$ s.t. $\mathcal{S}=(\xi,\eta,\Phi,g)\in(\mathcal{D},J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D},J)=\mathfrak{t}_k^+(\mathcal{D},J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D},J)$
  4. $\kappa(\mathcal{D},J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D},J)$
  5. $1\leq \dim\kappa(\mathcal{D},J)\leq n+1$ and if $\dim\kappa(\mathcal{D},J)=n+1$, M is **toric Sasakian**
- **Sasaki bouquets**
  1. a contact structure $\mathcal{D}$ of Sasaki type with a space of **compatible CR structures** $\mathcal{J}(\mathcal{D})$

Sasaki cones and bouquets

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D},J)=\{\xi'\in\mathfrak{t}_k\mid \eta(\xi')>0,\}$ s.t. $\mathcal{S}=(\xi,\eta,\Phi,g)\in(\mathcal{D},J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D},J)=\mathfrak{t}_k^+(\mathcal{D},J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D},J)$
  4. $\kappa(\mathcal{D},J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D},J)$
  5. $1\leq \dim\kappa(\mathcal{D},J)\leq n+1$ and if $\dim\kappa(\mathcal{D},J)=n+1$, M is **toric Sasakian**
- **Sasaki bouquets**
  1. a contact structure $\mathcal{D}$ of Sasaki type with a space of **compatible CR structures** $\mathcal{J}(\mathcal{D})$
  2. a map $\mathfrak{Q}:\mathcal{J}(\mathcal{D})\to\{$ **conjugacy classes of tori** in the contactomorphism group $\mathfrak{Con}(M,\mathcal{D})\}$

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D},J)=\{\xi'\in\mathfrak{t}_k \mid \eta(\xi')>0,\}$ s.t. $\mathcal{S}=(\xi,\eta,\Phi,g)\in(\mathcal{D},J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D},J)=\mathfrak{t}_k^+(\mathcal{D},J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D},J)$
  4. $\kappa(\mathcal{D},J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D},J)$
  5. $1\leq \dim\kappa(\mathcal{D},J)\leq n+1$ and if $\dim\kappa(\mathcal{D},J)=n+1$, M is **toric Sasakian**
- **Sasaki bouquets**
  1. a contact structure $\mathcal{D}$ of Sasaki type with a space of **compatible CR structures** $\mathcal{J}(\mathcal{D})$
  2. a map $\mathfrak{Q}:\mathcal{J}(\mathcal{D})\to\{$ **conjugacy classes of tori** in the contactomorphism group $\mathfrak{Con}(M,\mathcal{D})\}$
  3. Get **bouquet** $\bigcup_\alpha \kappa(\mathcal{D},J_\alpha)$ of Sasaki cones, $J_\alpha\in\mathcal{J}(\mathcal{D})$, α ranges over distinct conjugacy classes.

Sasaki cones and bouquets

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n + 1$ and if $\dim \kappa(\mathcal{D}, J) = n + 1$, M is **toric Sasakian**
- **Sasaki bouquets**
  1. a contact structure $\mathcal{D}$ of Sasaki type with a space of **compatible CR structures** $\mathcal{J}(\mathcal{D})$
  2. a map $\mathfrak{Q} : \mathcal{J}(\mathcal{D}) \rightarrow \{$ **conjugacy classes of tori** in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})\}$
  3. Get **bouquet** $\bigcup_\alpha \kappa(\mathcal{D}, J_\alpha)$ of Sasaki cones, $J_\alpha \in \mathcal{J}(\mathcal{D})$, α ranges over distinct conjugacy classes.
  4. A bouquet consisting of N Sasaki cones is called an **N-bouquet**, denoted by $\mathfrak{B}_N$. The Sasaki cones in an N-bouquet can have different dimension. The pre-moduli space is typically non-Hausdorff.

- **Sasaki cones**
  1. $\mathfrak{t}_k$ the Lie algebra of T^k
  2. **Sasaki cone** (unreduced): $\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi' \in \mathfrak{t}_k \mid \eta(\xi') > 0, \}$ s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ is Sasakian.
  3. **Sasaki cone** (reduced): $\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of $\mathfrak{CR}(\mathcal{D}, J)$
  4. $\kappa(\mathcal{D}, J)$ is finite dim'l **moduli of Sasakian structures** with underlying CR structure $(\mathcal{D}, J)$
  5. $1 \leq \dim \kappa(\mathcal{D}, J) \leq n + 1$ and if $\dim \kappa(\mathcal{D}, J) = n + 1$, M is **toric Sasakian**
- **Sasaki bouquets**
  1. a contact structure $\mathcal{D}$ of Sasaki type with a space of **compatible CR structures** $\mathcal{J}(\mathcal{D})$
  2. a map $\mathfrak{Q} : \mathcal{J}(\mathcal{D}) \to \{$ **conjugacy classes of tori** in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})\}$
  3. Get **bouquet** $\bigcup_\alpha \kappa(\mathcal{D}, J_\alpha)$ of Sasaki cones, $J_\alpha \in \mathcal{J}(\mathcal{D})$, α ranges over distinct conjugacy classes.
  4. A bouquet consisting of N Sasaki cones is called an **N-bouquet**, denoted by $\mathfrak{B}_N$. The Sasaki cones in an N-bouquet can have different dimension. The pre-moduli space is typically non-Hausdorff.
  5. the Sasaki cones $\kappa(\mathcal{D}, J_\alpha)$ can be distinguished by **equivariant Gromov-Witten invariants**

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
  1. First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.

- Three Types of **Deformations** of Sasakian Structures
  1. First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  2. After this type of deformation the transverse holonomy becomes irreducible.

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
  1. First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  2. After this type of deformation the transverse holonomy becomes irreducible.

  1. Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.

- Three Types of **Deformations** of Sasakian Structures
  1. First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  2. After this type of deformation the transverse holonomy becomes irreducible.

  1. Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.
  2. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})$.

- Three Types of **Deformations** of Sasakian Structures
  1. First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  2. After this type of deformation the transverse holonomy becomes irreducible.

  1. Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.
  2. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})$.

  1. Third type: Fix characteristic foliation $\mathcal{F}$, deform contact structure $\mathcal{D}$. This is used to obtain **extremal Sasaki metrics**.

Deformations of Sasakian Structures

- Three Types of **Deformations** of Sasakian Structures
  1. First type: Fix CR structure $(\mathcal{D}, J)$, deform characteristic foliation $\mathcal{F}$. This gives rise to **Sasaki cones**.
  2. After this type of deformation the transverse holonomy becomes irreducible.

  1. Second type: Fix contact structure $\mathcal{D}$, deform transverse complex structure (CR) J. This gives rise to **Sasaki bouquets**.
  2. Here Sasaki cones in bouquets are related to conjugacy classes of tori in the contactomorphism group $\mathfrak{Con}(M, \mathcal{D})$.

  1. Third type: Fix characteristic foliation $\mathcal{F}$, deform contact structure $\mathcal{D}$. This is used to obtain **extremal Sasaki metrics**.
  2. This type of deformation does not change the transverse holonomy nor the isotopy class of contact structure.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^\# s_g$ is transversely holomorphic.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^\# s_g$ is transversely holomorphic.
- We say that g is **extremal** if it is critical point of E.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic.
- We say that g is **extremal** if it is critical point of E.
- g is extremal Sasaki metric $\iff$ the transverse metric $g_{\mathcal{D}}$ is extremal Kähler metric.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic.
- We say that g is **extremal** if it is critical point of E.
- g is extremal Sasaki metric $\iff$ the transverse metric $g_{\mathcal{D}}$ is extremal Kähler metric.
- Special case: constant scalar curvature Sasakian **(CSC)**. If $c_1(\mathcal{D}) = 0 \Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get Sasaki-Einstein (SE).

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^\# s_g$ is transversely holomorphic.
- We say that g is **extremal** if it is critical point of E.
- g is extremal Sasaki metric $\iff$ the transverse metric $g_{\mathcal{D}}$ is extremal Kähler metric.
- Special case: constant scalar curvature Sasakian **(CSC)**. If $c_1(\mathcal{D}) = 0 \Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get Sasaki-Einstein (SE).
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is extremal (or CSC) then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g$,
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic.
- We say that g is **extremal** if it is critical point of E.
- g is extremal Sasaki metric $\iff$ the transverse metric $g_{\mathcal{D}}$ is extremal Kähler metric.
- Special case: constant scalar curvature Sasakian **(CSC)**. If $c_1(\mathcal{D}) = 0 \Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get Sasaki-Einstein (SE).
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is extremal (or CSC) then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.
- Calabi: Critical points have maximal symmetry.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g,$
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic.
- We say that g is **extremal** if it is critical point of E.
- g is extremal Sasaki metric $\iff$ the transverse metric $g_{\mathcal{D}}$ is extremal Kähler metric.
- Special case: constant scalar curvature Sasakian **(CSC)**. If $c_1(\mathcal{D}) = 0 \Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get Sasaki-Einstein (SE).
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is extremal (or CSC) then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.
- Calabi: Critical points have maximal symmetry.
- The Sasaki-Futaki invariant $\mathfrak{S}\mathfrak{F}(X) = \int_M X(\psi_g) d\mu_g$ where X is transversely holomorphic and ψ_g is the Ricci potential satisfying $\rho^T = \rho_h^T + i\partial\bar{\partial}\psi_g$ where ρ^T is the transverse Ricci form and ρ_h^T is its harmonic part.

Extremal Sasakian metrics (B-Galicki-Simanca)

- Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with scalar curvature s_g.
- Calabi-Sasaki Energy functional $E(g) = \int_M s_g^2 d\mu_g$,
- Deform contact structure $\eta \mapsto \eta + td^c\varphi$ within its isotopy class where φ is basic.
- This gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic.
- We say that g is **extremal** if it is critical point of E.
- g is extremal Sasaki metric $\iff$ the transverse metric $g_{\mathcal{D}}$ is extremal Kähler metric.
- Special case: constant scalar curvature Sasakian **(CSC)**. If $c_1(\mathcal{D}) = 0 \Rightarrow$ Sasaki-η-Einstein (SηE) with Ricci curvature $\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants. If $b = 0$ get Sasaki-Einstein (SE).
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is extremal (or CSC) then so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for any $a > 0$.
- Calabi: Critical points have maximal symmetry.
- The Sasaki-Futaki invariant $\mathfrak{S}\mathfrak{F}(X) = \int_M X(\psi_g) d\mu_g$ where X is transversely holomorphic and ψ_g is the Ricci potential satisfying $\rho^T = \rho_h^T + i\partial\bar{\partial}\psi_g$ where ρ^T is the transverse Ricci form and ρ_h^T is its harmonic part.

Theorem

An extremal Sasaki metric g has **constant scalar curvature** *if and only if $\mathfrak{S}\mathfrak{F} = 0$.*

The Join Construction

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.

The Join Construction

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
- The dimension of $M_1 \star_{l_1,l_2} M_2$ is $n_1 + n_2 - 1$.

The Join Construction

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
- The dimension of $M_1 \star_{l_1,l_2} M_2$ is $n_1 + n_2 - 1$.
- The join $M_1 \star_{l_1,l_2} M_2$ has **reducible transverse holonomy** a subgroup of $U(n_1) \times U(n_2)$.

The Join Construction

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
- The dimension of $M_1 \star_{l_1,l_2} M_2$ is $n_1 + n_2 - 1$.
- The join $M_1 \star_{l_1,l_2} M_2$ has **reducible transverse holonomy** a subgroup of $U(n_1) \times U(n_2)$.
- Take $\pi_2 : M_2 \longrightarrow \mathcal{Z}_2$ to be the S^1 orbibundle $\pi_2 : S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1[\mathbf{w}]$ determined by a weighted S^1 action on S^3 with weights $\mathbf{w} = (w_1, w_2)$ satisfying $\gcd(l_2, l_1 w_i) = 1$, and $M_1 = M$ regular Sasaki manifold whose quotient is a compact Kähler manifold N.

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
- The dimension of $M_1 \star_{l_1,l_2} M_2$ is $n_1 + n_2 - 1$.
- The join $M_1 \star_{l_1,l_2} M_2$ has **reducible transverse holonomy** a subgroup of $U(n_1) \times U(n_2)$.
- Take $\pi_2 : M_2 \longrightarrow \mathcal{Z}_2$ to be the S^1 orbibundle $\pi_2 : S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1[\mathbf{w}]$ determined by a weighted S^1 action on S^3 with weights $\mathbf{w} = (w_1, w_2)$ satisfying $\gcd(l_2, l_1 w_i) = 1$, and $M_1 = M$ regular Sasaki manifold whose quotient is a compact Kähler manifold N.
- In this case the **Join Construction** and **Admissible Contruction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman fit as hand and glove.

The Join Construction

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
- The dimension of $M_1 \star_{l_1,l_2} M_2$ is $n_1 + n_2 - 1$.
- The join $M_1 \star_{l_1,l_2} M_2$ has **reducible transverse holonomy** a subgroup of $U(n_1) \times U(n_2)$.
- Take $\pi_2 : M_2 \longrightarrow \mathcal{Z}_2$ to be the S^1 orbibundle $\pi_2 : S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1[\mathbf{w}]$ determined by a weighted S^1 action on S^3 with weights $\mathbf{w} = (w_1, w_2)$ satisfying $\gcd(l_2, l_1 w_i) = 1$, and $M_1 = M$ regular Sasaki manifold whose quotient is a compact Kähler manifold N.
- In this case the **Join Construction** and **Admissible Contruction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman fit as hand and glove.
- The transverse holonomy of $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ is generically $U(n_1) \times U(2)$.

The Join Construction

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
- The dimension of $M_1 \star_{l_1,l_2} M_2$ is $n_1 + n_2 - 1$.
- The join $M_1 \star_{l_1,l_2} M_2$ has **reducible transverse holonomy** a subgroup of $U(n_1) \times U(n_2)$.
- Take $\pi_2 : M_2 \longrightarrow \mathcal{Z}_2$ to be the S^1 orbibundle $\pi_2 : S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1[\mathbf{w}]$ determined by a weighted S^1 action on S^3 with weights $\mathbf{w} = (w_1, w_2)$ satisfying $\gcd(l_2, l_1 w_i) = 1$, and $M_1 = M$ regular Sasaki manifold whose quotient is a compact Kähler manifold N.
- In this case the **Join Construction** and **Admissible Contruction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman fit as hand and glove.
- The transverse holonomy of $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ is generically $U(n_1) \times U(2)$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Kähler.

The Join Construction

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
- The dimension of $M_1 \star_{l_1,l_2} M_2$ is $n_1 + n_2 - 1$.
- The join $M_1 \star_{l_1,l_2} M_2$ has **reducible transverse holonomy** a subgroup of $U(n_1) \times U(n_2)$.
- Take $\pi_2 : M_2 \longrightarrow \mathcal{Z}_2$ to be the S^1 orbibundle $\pi_2 : S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1[\mathbf{w}]$ determined by a weighted S^1 action on S^3 with weights $\mathbf{w} = (w_1, w_2)$ satisfying $\gcd(l_2, l_1 w_i) = 1$, and $M_1 = M$ regular Sasaki manifold whose quotient is a compact Kähler manifold N.
- In this case the **Join Construction** and **Admissible Contruction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman fit as hand and glove.
- The transverse holonomy of $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ is generically $U(n_1) \times U(2)$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Kähler.
- The join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ can be realized as a lens space bundle over N with fiber the lens space $L(l_2; l_1 w_1, l_1, w_2)$.

The Join Construction

- **Join Construction**: Given quasi-regular Sasakian manifolds $\pi_i : M_i \longrightarrow \mathcal{Z}_i$ with $\mathrm{Dim}\, M_i = n_i$ for $i = 1, 2$.
- Form (l_1, l_2)-join $\pi : M_1 \star_{l_1,l_2} M_2 \longrightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$ as an S^1-orbibundle (B-,Galicki,Ornea).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
- The dimension of $M_1 \star_{l_1,l_2} M_2$ is $n_1 + n_2 - 1$.
- The join $M_1 \star_{l_1,l_2} M_2$ has **reducible transverse holonomy** a subgroup of $U(n_1) \times U(n_2)$.
- Take $\pi_2 : M_2 \longrightarrow \mathcal{Z}_2$ to be the S^1 orbibundle $\pi_2 : S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1[\mathbf{w}]$ determined by a weighted S^1 action on S^3 with weights $\mathbf{w} = (w_1, w_2)$ satisfying $\gcd(l_2, l_1 w_i) = 1$, and $M_1 = M$ regular Sasaki manifold whose quotient is a compact Kähler manifold N.
- In this case the **Join Construction** and **Admissible Contruction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman fit as hand and glove.
- The transverse holonomy of $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ is generically $U(n_1) \times U(2)$.
- An S^1 orbibundle $M \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow N \times \mathbb{CP}^1[\mathbf{w}]$, where N is compact Kähler.
- The join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ can be realized as a lens space bundle over N with fiber the lens space $L(l_2; l_1 w_1, l_1, w_2)$.
- I present two **fundamental theorems** about $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ and then present brief outlines of their proofs. Finally, I discuss the special case of S^3-bundles over a Riemann surface Σ_g.

Fundamental Theorem: Geometry

- Geometry: Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

Fundamental Theorem: Geometry

- Geometry: Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

Theorem (1)

*[B-,Tønnesen-Friedman]: Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-**join** with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in a 2-dimensional sub cone, the $\mathbf{w}$-cone, of the Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has **constant scalar curvature**.*

Fundamental Theorem: Geometry

- Geometry: Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

Theorem (1)

*[B-,Tønnesen-Friedman]: Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-**join** with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in a 2-dimensional sub cone, the $\mathbf{w}$-cone, of the Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has* **constant scalar curvature**.

1. *If the scalar curvature s_N of N is nonnegative, then the $\mathbf{w}$-cone is exhausted by extremal Sasaki metrics.*

Fundamental Theorem: Geometry

- Geometry: Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

Theorem (1)

*[B-,Tønnesen-Friedman]: Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-**join** with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in a 2-dimensional sub cone, the $\mathbf{w}$-cone, of the Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has* **constant scalar curvature**.

1. *If the scalar curvature s_N of N is nonnegative, then the $\mathbf{w}$-cone is exhausted by extremal Sasaki metrics.*
2. *If the scalar curvature s_N of N is positive and l_2 is large enough there are infinitely many contact CR structures with at least 3 rays of* **CSC** *Sasakian structures in the $\mathbf{w}$-cone.*

Fundamental Theorem: Geometry

- Geometry: Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

Theorem (1)

*[B-,Tønnesen-Friedman]: Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-**join** with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in a 2-dimensional sub cone, the $\mathbf{w}$-cone, of the Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has* **constant scalar curvature**.

1. *If the scalar curvature s_N of N is nonnegative, then the $\mathbf{w}$-cone is exhausted by extremal Sasaki metrics.*
2. *If the scalar curvature s_N of N is positive and l_2 is large enough there are infinitely many contact CR structures with at least 3 rays of* **CSC** *Sasakian structures in the $\mathbf{w}$-cone.*
3. *When N is positive* **KE** *get* **SE** *metric on $M_{l_1,l_2,\mathbf{w}}$ for appropriate choice of (l_1, l_2).*

Fundamental Theorem: Geometry

- Geometry: Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

Theorem (1)

*[B-,Tønnesen-Friedman]: Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-**join** with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in a 2-dimensional sub cone, the $\mathbf{w}$-cone, of the Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has* **constant scalar curvature**.

1. *If the scalar curvature s_N of N is nonnegative, then the $\mathbf{w}$-cone is exhausted by extremal Sasaki metrics.*
2. *If the scalar curvature s_N of N is positive and l_2 is large enough there are infinitely many contact CR structures with at least 3 rays of* **CSC** *Sasakian structures in the $\mathbf{w}$-cone.*
3. *When N is positive* **KE** *get* **SE** *metric on $M_{l_1,l_2,\mathbf{w}}$ for appropriate choice of (l_1, l_2).*

- The SE metrics of 3 were previously obtained by physicists (Gauntlett, Martelli, Sparks, Waldram) by another method.

Fundamental Theorem: Geometry

- Geometry: Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

Theorem (1)

*[B-,Tønnesen-Friedman]: Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-**join** with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in a 2-dimensional sub cone, the $\mathbf{w}$-cone, of the Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has* **constant scalar curvature**.

1. *If the scalar curvature s_N of N is nonnegative, then the $\mathbf{w}$-cone is exhausted by extremal Sasaki metrics.*
2. *If the scalar curvature s_N of N is positive and l_2 is large enough there are infinitely many contact CR structures with at least 3 rays of* **CSC** *Sasakian structures in the $\mathbf{w}$-cone.*
3. *When N is positive* **KE** *get* **SE** *metric on $M_{l_1,l_2,\mathbf{w}}$ for appropriate choice of (l_1, l_2).*

- The SE metrics of 3 were previously obtained by physicists (Gauntlett, Martelli, Sparks, Waldram) by another method.
- Most of the **CSC** Sasakian structures are **irregular**.

Fundamental Theorem: Geometry

- Geometry: Existence of **extremal** and **CSC** Sasaki metrics by deforming in the Sasaki cone

Theorem (1)

*[B-,Tønnesen-Friedman]: Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-**join** with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in a 2-dimensional sub cone, the $\mathbf{w}$-cone, of the Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has* **constant scalar curvature***.*

1. *If the scalar curvature s_N of N is nonnegative, then the $\mathbf{w}$-cone is exhausted by extremal Sasaki metrics.*
2. *If the scalar curvature s_N of N is positive and l_2 is large enough there are infinitely many contact CR structures with at least 3 rays of* **CSC** *Sasakian structures in the $\mathbf{w}$-cone.*
3. *When N is positive* **KE** *get* **SE** *metric on $M_{l_1,l_2,\mathbf{w}}$ for appropriate choice of (l_1, l_2).*

- The SE metrics of 3 were previously obtained by physicists (Gauntlett, Martelli, Sparks, Waldram) by another method.
- Most of the **CSC** Sasakian structures are **irregular**.
- Relation to **CR Yamabe Problem** (Jerison and Lee): For a Sasaki structure the Webster pseudo-Hermitian metric coincides with the transverse Kähler metric. So a **CSC** Sasaki metric provides a solution to the CR Yamabe Problem. It is know that when the **CR Yamabe invariant** $\lambda(M)$ is **nonpositive**, the CSC metric is unique. However, when $\lambda(M) > 0$ there can be several CSC solutions. Our results provides many such examples.

Fundamental Theorem: Topology

- Topology: given the cohomology ring of M, the **cohomology ring** of $M_{l_1,l_2,\mathbf{w}}$ can be determined in principal.

Fundamental Theorem: Topology

- Topology: given the cohomology ring of M, the **cohomology ring** of $M_{l_1,l_2,\mathbf{w}}$ can be determined in principal.

Theorem (2)

[B-,Tønnesen-Friedman]: In each odd dimension $2p+3>5$ there exist countably infinite simply connected **toric contact** *manifolds $M_{l_1,l_2,\mathbf{w}}$ of Reeb type depending on 4 positive integers l_1,l_2,w_1,w_2 satisfying $\gcd(l_2,l_1w_i)=\gcd(w_1,w_2)=1$, and with* **integral cohomology ring**

$$H^*(M_{l_1,l_2,\mathbf{w}},\mathbb{Z})\approx\mathbb{Z}[x,y]/(w_1w_2l_1^2x^2,x^{p+1},x^2y,y^2)$$

where x,y are classes of degree 2 and $2p+1$, respectively. Furthermore, with l_1,w_1,w_2 fixed there are a **finite number of diffeomorphism types** *with the given cohomology ring. Hence, in each such dimension there exist simply connected smooth manifolds with* **countably infinite toric contact** *structures of Reeb type that are* **inequivalent** *as contact structures.*

Fundamental Theorem: Topology

- Topology: given the cohomology ring of M, the **cohomology ring** of $M_{l_1,l_2,\mathbf{w}}$ can be determined in principal.

Theorem (2)

[B-,Tønnesen-Friedman]: In each odd dimension $2p+3>5$ there exist countably infinite simply connected **toric contact** *manifolds $M_{l_1,l_2,\mathbf{w}}$ of Reeb type depending on 4 positive integers l_1, l_2, w_1, w_2 satisfying $\gcd(l_2, l_1 w_i) = \gcd(w_1, w_2) = 1$, and with* **integral cohomology ring**

$$H^*(M_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) \approx \mathbb{Z}[x,y]/(w_1 w_2 l_1^2 x^2, x^{p+1}, x^2 y, y^2)$$

where x, y are classes of degree 2 and $2p+1$, respectively. Furthermore, with l_1, w_1, w_2 fixed there are a **finite number of diffeomorphism types** *with the given cohomology ring. Hence, in each such dimension there exist simply connected smooth manifolds with* **countably infinite toric contact** *structures of Reeb type that are* **inequivalent** *as contact structures.*

- In special cases we can determine the **diffeomorphism (homeomorphism, homotopy)** types.

Outline of proof of Theorem (1):

- The existence of an extra Hamiltonian Killing vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.

Outline of proof of Theorem (1):

- The existence of an extra Hamiltonian Killing vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.

Outline of proof of Theorem (1):

- The existence of an extra Hamiltonian Killing vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

  consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$

Outline of proof of Theorem (1):

- The existence of an extra Hamiltonian Killing vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

  consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.

Outline of proof of Theorem (1):

- The existence of an extra Hamiltonian Killing vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

  consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})^d$ is a $(d+3)$ order ($(d+2)$ order) **polynomial** we get **extremal (CSC) Kähler metrics**. Here d is the complex dimension of N.

Outline of proof of Theorem (1):

- The existence of an extra Hamiltonian Killing vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

  consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})^d$ is a $(d+3)$ order ($(d+2)$ order) **polynomial** we get **extremal (CSC) Kähler metrics**. Here d is the complex dimension of N.
- Lifing to $M_{l_1,l_2,\mathbf{w}}$ gives **extremal (CSC) Sasaki metrics** in the quasi-regular case.

Outline of proof of Theorem (1):

- The existence of an extra Hamiltonian Killing vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})^d$ is a $(d + 3)$ order ($(d + 2)$ order) **polynomial** we get **extremal (CSC) Kähler metrics**. Here d is the complex dimension of N.
- Lifing to $M_{l_1,l_2,\mathbf{w}}$ gives **extremal (CSC) Sasaki metrics** in the quasi-regular case.
- The **irregular** case uses a continuity argument together with the fact that quasi-regular Sasaki structures are dense in the Sasaki cone.

Outline of proof of Theorem (1):

- The existence of an extra Hamiltonian Killing vector field from $S^3_{\mathbf{w}}$ gives the 2-dimensional Sasaki $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$.
- The quotient space of the S^1-action generated by any quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is a ruled orbifold $(S_n, \Delta_{mv_1,mv_2})$ with a branch divisor

$$\Delta_{mv_1,mv_2} = (1 - \frac{1}{mv_1})D_1 + (1 - \frac{1}{mv_2})D_2$$

  consisting of the zero D_1 and infinity D_2 sections of the **projective bundle** $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ over N with ramification indices mv_1, mv_2, respectively and n an integer determined by $l_1, l_2, \mathbf{w}, \mathbf{v}$.
- For $n \neq 0$, apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds $(S_n, \Delta_{mv_1,mv_2})$
- This gives the Kähler orbifold metric $g_{(S_n,\Delta)} = \frac{1+r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$ where θ is a connection 1-form, $d\theta = n\omega_N$, $0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ and $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(-1) = \frac{2}{mv_2}, \Theta'(1) = -\frac{2}{mv_1}$.
- When $\Theta(\mathfrak{z})(1 + r\mathfrak{z})^d$ is a $(d+3)$ order ($(d+2)$ order) **polynomial** we get **extremal (CSC) Kähler metrics**. Here d is the complex dimension of N.
- Lifing to $M_{l_1,l_2,\mathbf{w}}$ gives **extremal (CSC) Sasaki metrics** in the quasi-regular case.
- The **irregular** case uses a continuity argument together with the fact that quasi-regular Sasaki structures are dense in the Sasaki cone.
- The existence of multiple rays of **CSC** Sasaki metrics comes from a sign changing count.

Outline of proof of Theorem (2):

- From join construction get commutative diagram of fibrations:

$$\begin{array}{ccccc}
M \times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathrm{B}S^1 \\
\Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \psi} \\
M \times S^3_{\mathbf{w}} & \longrightarrow & N \times \mathrm{B}\mathbb{C}\mathbb{P}^1[\mathbf{w}] & \longrightarrow & \mathrm{B}S^1 \times \mathrm{B}S^1
\end{array}$$

where $\mathrm{B}G$ is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.

Outline of proof of Theorem (2):

- From join construction get commutative diagram of fibrations:

$$\begin{array}{ccccc} M\times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathsf{B}S^1 \\ \Big\downarrow = & & \Big\downarrow & & \Big\downarrow \psi \\ M\times S^3_{\mathbf{w}} & \longrightarrow & N\times \mathsf{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathsf{B}S^1\times \mathsf{B}S^1 \end{array}$$

  where $\mathsf{B}G$ is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.
- Given the differentials in the **spectral sequence** of the fibration $M\longrightarrow N\longrightarrow \mathsf{B}S^1$, use the commutative diagram to compute the **cohomology ring** of the **contact** manifold $M_{l_1,l_2,\mathbf{w}}$ with

Outline of proof of Theorem (2):

- From join construction get commutative diagram of fibrations:

$$\begin{array}{ccccc} M \times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathrm{B}S^1 \\ \Big\downarrow = & & \Big\downarrow & & \Big\downarrow \psi \\ M \times S^3_{\mathbf{w}} & \longrightarrow & N \times \mathrm{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathrm{B}S^1 \times \mathrm{B}S^1 \end{array}$$

  where $\mathrm{B}G$ is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.
- Given the differentials in the **spectral sequence** of the fibration $M \longrightarrow N \longrightarrow \mathrm{B}S^1$, use the commutative diagram to compute the **cohomology ring** of the **contact** manifold $M_{l_1,l_2,\mathbf{w}}$ with
- orbifold cohomology:

$$H^r_{orb}(\mathbb{CP}^1[\mathbf{w}],\mathbb{Z}) = H^r(\mathrm{B}\mathbb{CP}^1[\mathbf{w}],\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } r = 0,2, \\ \mathbb{Z}_{w_1 w_2} & \text{for } r > 2 \text{ even}, \\ 0 & \text{for } r \text{ odd}. \end{cases}$$

Outline of proof of Theorem (2):

- From join construction get commutative diagram of fibrations:

$$\begin{array}{ccccc} M\times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathrm{B}S^1 \\ \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\psi} \\ M\times S^3_{\mathbf{w}} & \longrightarrow & N\times \mathrm{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathrm{B}S^1\times \mathrm{B}S^1 \end{array}$$

  where $\mathrm{B}G$ is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.
- Given the differentials in the **spectral sequence** of the fibration $M\longrightarrow N\longrightarrow \mathrm{B}S^1$, use the commutative diagram to compute the **cohomology ring** of the **contact** manifold $M_{l_1,l_2,\mathbf{w}}$ with
- orbifold cohomology:

$$H^r_{orb}(\mathbb{CP}^1[\mathbf{w}],\mathbb{Z})=H^r(\mathrm{B}\mathbb{CP}^1[\mathbf{w}],\mathbb{Z})=\begin{cases}\mathbb{Z} & \text{for } r=0,2,\\ \mathbb{Z}_{w_1w_2} & \text{for } r>2 \text{ even},\\ 0 & \text{for } r \text{ odd}.\end{cases}$$

- For Theorem (2) we take the sphere $M=S^{2p+1}$ with $p>1$ in which case the cohomology ring follows by computing the differentials. $p=1$ case later (S^3-bundles over S^2).

Outline of proof of Theorem (2):

- From join construction get commutative diagram of fibrations:

$$\begin{array}{ccccc} M \times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathrm{B}S^1 \\ \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\psi} \\ M \times S^3_{\mathbf{w}} & \longrightarrow & N \times \mathrm{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathrm{B}S^1 \times \mathrm{B}S^1 \end{array}$$

  where $\mathrm{B}G$ is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.
- Given the differentials in the **spectral sequence** of the fibration $M \longrightarrow N \longrightarrow \mathrm{B}S^1$, use the commutative diagram to compute the **cohomology ring** of the **contact** manifold $M_{l_1,l_2,\mathbf{w}}$ with
- orbifold cohomology:

$$H^r_{orb}(\mathbb{CP}^1[\mathbf{w}],\mathbb{Z}) = H^r(\mathrm{B}\mathbb{CP}^1[\mathbf{w}],\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } r = 0,2, \\ \mathbb{Z}_{w_1 w_2} & \text{for } r > 2 \text{ even}, \\ 0 & \text{for } r \text{ odd}. \end{cases}$$

- For Theorem (2) we take the sphere $M = S^{2p+1}$ with $p > 1$ in which case the cohomology ring follows by computing the differentials. $p = 1$ case later (S^3-bundles over S^2).
- The finiteness of the **diffeomorphism types** follows by Sullivan's rational homotopy theory. $M_{l_1,l_2,\mathbf{w}}$ is **formal**.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
- When $g > 0$ set $l_2 = 1$ (B-,Tonnesen-Friedman), S^3-bundles over a **Riemann surface** Σ_g with two **diffeomorphism types**, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
- When $g > 0$ set $l_2 = 1$ (B-,Tonnesen-Friedman), S^3-bundles over a **Riemann surface** Σ_g with two **diffeomorphism types**, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.
- On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a ray of **CSC** Sasakian structures.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
- When $g > 0$ set $l_2 = 1$ (B-,Tonnesen-Friedman), S^3-bundles over a **Riemann surface** Σ_g with two **diffeomorphism types**, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.
- On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a ray of **CSC** Sasakian structures.
- When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(\mathcal{D}_k, J)$ on S^3-bundles over Σ_g are exhausted by **extremal Sasaki metrics**

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
- When $g > 0$ set $l_2 = 1$ (B-,Tonnesen-Friedman), S^3-bundles over a **Riemann surface** Σ_g with two **diffeomorphism types**, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.
- On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a ray of **CSC** Sasakian structures.
- When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(\mathcal{D}_k, J)$ on S^3-bundles over Σ_g are exhausted by **extremal Sasaki metrics**
- For $g \geq 20$ there are rays in $\kappa(\mathcal{D}_k, J)$ which admit **no** extremal Sasaki metrics.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
- When $g > 0$ set $l_2 = 1$ (B-,Tonnesen-Friedman), S^3-bundles over a **Riemann surface** Σ_g with two **diffeomorphism types**, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.
- On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a ray of **CSC** Sasakian structures.
- When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(\mathcal{D}_k, J)$ on S^3-bundles over Σ_g are exhausted by **extremal Sasaki metrics**
- For $g \geq 20$ there are rays in $\kappa(\mathcal{D}_k, J)$ which admit **no** extremal Sasaki metrics.
- For any genus $g \geq 1$ and for each positive integer k, the contact manifold $(\Sigma_g \times S^3, \mathcal{D}_k)$ has a k-**bouquet** $\mathfrak{B}_k$ of 2-dimensional Sasaki cones.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
- When $g > 0$ set $l_2 = 1$ (B-,Tonnesen-Friedman), S^3-bundles over a **Riemann surface** Σ_g with two **diffeomorphism types**, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.
- On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a ray of **CSC** Sasakian structures.
- When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(\mathcal{D}_k, J)$ on S^3-bundles over Σ_g are exhausted by **extremal Sasaki metrics**
- For $g \geq 20$ there are rays in $\kappa(\mathcal{D}_k, J)$ which admit **no** extremal Sasaki metrics.
- For any genus $g \geq 1$ and for each positive integer k, the contact manifold $(\Sigma_g \times S^3, \mathcal{D}_k)$ has a k-**bouquet** $\mathfrak{B}_k$ of 2-dimensional Sasaki cones.
- The distinct Sasaki cones in the bouquet $\mathfrak{B}_k$ correspond to distinct conjugacy classes of maximal tori in $\mathfrak{Con}(\mathcal{D}_{l_1,l_2,\mathbf{w}})$. Uses the work of Buşe on equivariant Gromov-Witten invariants.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
- When $g > 0$ set $l_2 = 1$ (B-,Tonnesen-Friedman), S^3-bundles over a **Riemann surface** Σ_g with two **diffeomorphism types**, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.
- On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a ray of **CSC** Sasakian structures.
- When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(\mathcal{D}_k, J)$ on S^3-bundles over Σ_g are exhausted by **extremal Sasaki metrics**
- For $g \geq 20$ there are rays in $\kappa(\mathcal{D}_k, J)$ which admit **no** extremal Sasaki metrics.
- For any genus $g \geq 1$ and for each positive integer k, the contact manifold $(\Sigma_g \times S^3, \mathcal{D}_k)$ has a k-**bouquet** $\mathfrak{B}_k$ of 2-dimensional Sasaki cones.
- The distinct Sasaki cones in the bouquet $\mathfrak{B}_k$ correspond to distinct conjugacy classes of maximal tori in $\mathfrak{Con}(\mathcal{D}_{l_1,l_2,\mathbf{w}})$. Uses the work of Buşe on equivariant Gromov-Witten invariants.
- The construction can be 'twisted' by reducible representations of the fundamental group $\pi_1(\Sigma_g)$. The irreducible representations of $\pi_1(\Sigma_g)$ give 1-dimensional Sasaki cones. They arise from stable rank two vector bundles and have CSC Sasaki metrics.

S^3-bundles over Riemann surfaces

- Case 1: $\dim N = 1$. Take $N = \Sigma_g$, a compact **Riemann surface** of genus g.
- When $g = 0$ we get Sasakian structures on the two S^3-bundles over the S^2 for all relatively prime positive integers l_1, l_2. (B-,B-Pati) (Also E. Legendre). When $c_1 = 0$ we recover the **SE** metrics on $Y^{p,q}$ of the physicists Guantlett, Martelli, Sparks, Waldram. For fixed p the $\phi(p)$ (Euler phi-function) inequivalent **SE** structures belong to the same contact structure, a $\phi(p)$-**bouquet** $\mathfrak{B}_{\phi(p)}$.
- When $g > 0$ set $l_2 = 1$ (B-,Tonnesen-Friedman), S^3-bundles over a **Riemann surface** Σ_g with two **diffeomorphism types**, the trivial bundle $\Sigma_g \times S^3$, the non-trivial bundle $\Sigma_g \tilde{\times} S^3$.
- On both manifolds there is a countably infinite number of inequivalent contact structures $\mathcal{D}_k$ admitting a 2-dimensional cone of Sasakian structures which by our Fundamental Theorem 1 admits a ray of **CSC** Sasakian structures.
- When $0 < g \leq 4$ all 2-dimensional Sasaki cones $\kappa(\mathcal{D}_k, J)$ on S^3-bundles over Σ_g are exhausted by **extremal Sasaki metrics**
- For $g \geq 20$ there are rays in $\kappa(\mathcal{D}_k, J)$ which admit **no** extremal Sasaki metrics.
- For any genus $g \geq 1$ and for each positive integer k, the contact manifold $(\Sigma_g \times S^3, \mathcal{D}_k)$ has a k-**bouquet** $\mathfrak{B}_k$ of 2-dimensional Sasaki cones.
- The distinct Sasaki cones in the bouquet $\mathfrak{B}_k$ correspond to distinct conjugacy classes of maximal tori in $\mathfrak{Con}(\mathcal{D}_{l_1,l_2,\mathbf{w}})$. Uses the work of Buşe on equivariant Gromov-Witten invariants.
- The construction can be 'twisted' by reducible representations of the fundamental group $\pi_1(\Sigma_g)$. The irreducible representations of $\pi_1(\Sigma_g)$ give 1-dimensional Sasaki cones. They arise from stable rank two vector bundles and have CSC Sasaki metrics.
- Some of the same type of results have been obtained on 5-manifolds whose fundamental group is a non-Abelian extension of $\pi_1(\Sigma_g)$ in Castañeda's thesis.

THANK YOU!

References

1. Extremal Sasakian Metrics on S^3-bundles over S^2, Math. Res. Lett. 18 (2011), no. 01, 181-189.
2. Maximal Tori in Contactomorphism Groups, Diff. Geom and Appl. 21, no. 2 (2013), 190-216 (Math arXiv:1003.1903).
3. Completely Integrable Contact Hamiltonian Systems and Toric Contact Structures on $S^2 \times S^3$, SIGMA Symmetry Integrability Geom. Methods Appl 7 (2011),058, 22.
4. (with J. Pati) On the Equivalence Problem for Toric Contact Structures on S^3-bundles over S^2 Pacific J. Math. 267 (2) (2014), 277-324. (Math arXiv:1204.2209).
5. (with C. Tønnesen-Friedman) Extremal Sasakian Geometry on $T^2 \times S^3$ and Related Manifolds Compositio Math. 149 (2013), 1431-1456 (Math arXiv:1108.2005).
6 (with C. Tønnesen-Friedman) Sasakian Manifolds with Perfect Fundamental Groups, African Diaspora Journal of Mathematics 14 (2) (2012), 98-117.
7. (with C. Tønnesen-Friedman) Extremal Sasakian Geometry on S^3-bundles over Riemann Surfaces (to appear in International Mathematical Research Notices) Math ArXiv:1302.0776.
8. (with C. Tønnesen-Friedman) The Sasaki Join, Hamiltonian 2-forms, and Sasaki-Einstein Metrics (submitted for publication) Math ArXiv:1309.7067.
9. (with C. Tønnesen-Friedman) The Sasaki Join, Hamiltonian 2-forms, and Constant Scalar Curvature (submitted for publication) Math ArXiv:1402.2546.
10. (with C. Tønnesen-Friedman) Simply Connected Manifolds with Infinitely many Toric Contact Structures and Constant Scalar Curvature Sasaki Metrics (submitted for publication) Math ArXiv:1404.3999 .

General Reference: **C.P. B- and K. Galicki, Sasakian Geometry**, Oxford University Press, 2008.