# Extremal Sasakian Geometry:

the Join and Admissible Constructions

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Talk based on joint work with Christina Tønnesen-Friedman

Problems: (1) Given a manifold determine how many contact structures of Sasaki type there are.

(2) Given a contact structure or isotopy class of contact structures:

Determine the space of compatible Sasakian structures.

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- We give partial answers to these problems for particular cases that are obtained by combining the Sasaki join construction of B-, Galicki, Ornea with the admissible Kähler construction of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman.

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• The pair  $(\mathfrak{D}, J)$  is a **strictly pseudo-convex almost CR structure** (s $\psi$ CR structure).

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The contact metric structure  $\mathcal{S}=(\xi,\eta,\Phi,g)$  is **K-contact** if  $\mathcal{L}_{\xi}g=0$  (or  $\mathcal{L}_{\xi}\Phi=0$ ). It is **Sasakian** if in addition  $(\mathcal{D},J)$  is integrable and the **Transverse Metric**  $g_{\mathcal{D}}$  is Kähler (**Transverse holonomy** U(n)). In the latter case we say that the contact structure  $\mathcal{D}$  is of **Sasaki type**.

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- maximal torus:  $T^k \subset \mathfrak{Aut}(S) \subset \mathfrak{CR}(\mathcal{D}, J) \subset \mathfrak{Con}(M, \mathcal{D})$  with  $1 \leq k \leq n+1$

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## **Deformations of Sasakian Structures**

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  - Third type: Fix characteristic foliation F, deform contact structure D. This is used to obtain extremal Sasaki metrics.
  - This type of deformation does not change the transverse holonomy nor the isotopy class of contact structure.

### Extremal Sasakian metrics (B-Galicki-Simanca)

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- We say that g is extremal if it is critical point of E.

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### Extremal Sasakian metrics (B-Galicki-Simanca)

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### **Theorem**

An extremal Sasaki metric g has constant scalar curvature if and only if  $\mathfrak{S}\mathfrak{F}=0$ .

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- I present two fundamental theorems about M \*h, l<sub>2</sub> S<sup>3</sup><sub>w</sub> and then present brief outlines of their proofs. Finally, I discuss the special case of S<sup>3</sup>-bundles over a Riemann surface Σ<sub>g</sub>.

• Geometry: Existence of extremal and CSC Sasaki metrics by deforming in the Sasaki cone

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- If the scalar curvature s<sub>N</sub> of N is positive and l<sub>2</sub> is large enough there are infinitely many contact CR structures with at least 3 rays of CSC Sasakian structures in the w-cone.

Geometry: Existence of extremal and CSC Sasaki metrics by deforming in the Sasaki cone

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[B-, Tønnesen-Friedman]: Let  $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S_{\mathbf{w}}^3$  be the  $S_{\mathbf{w}}^3$ -join with a regular Sasaki manifold M which is an  $S^1$ -bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector  $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$  with relatively prime components satisfying  $w_1 > w_2$  there exists a Reeb vector field  $\xi_{\mathbf{v}}$  in a 2-dimensional sub cone, the  $\mathbf{w}$ -cone, of the Sasaki cone on  $M_{l_1,l_2,\mathbf{w}}$  such that the corresponding ray of Sasakian structures  $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}},a\eta_{\mathbf{v}},\Phi,g_a)$  has constant scalar curvature.

- If the scalar curvature s<sub>N</sub> of N is nonnegative, then the w-cone is exhausted by extremal Sasaki metrics.
- If the scalar curvature s<sub>N</sub> of N is positive and l<sub>2</sub> is large enough there are infinitely many contact CR structures with at least 3 rays of CSC Sasakian structures in the w-cone.
- **3** When N is positive **KE** get **SE** metric on  $M_{l_1, l_2, \mathbf{w}}$  for appropriate choice of  $(l_1, l_2)$ .

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  - Most of the CSC Sasakian structures are irregular.
  - Relation to CR Yamabe Problem (Jerison and Lee): For a Sasaki structure the Webster pseudo-Hermitian metric coincides with the transverse Kähler metric. So a CSC Sasaki metric provides a solution to the CR Yamabe Problem. It is know that when the CR Yamabe invariant  $\lambda(M)$  is nonpositive, the CSC metric is unique. However, when  $\lambda(M) > 0$  there can be several CSC solutions. Our results provides many such examples.

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$$H^*(M_{l_1,l_2,\mathbf{w}},\mathbb{Z}) \approx \mathbb{Z}[x,y]/(w_1w_2l_1^2x^2,x^{p+1},x^2y,y^2)$$

where x,y are classes of degree 2 and 2p+1, respectively. Furthermore, with  $l_1, w_1, w_2$  fixed there are a finite number of diffeomorphism types with the given cohomology ring. Hence, in each such dimension there exist simply connected smooth manifolds with countably infinite toric contact structures of Reeb type that are inequivalent as contact structures.

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 In special cases we can determine the diffeomorphism (homeomorphism, homotopy) types.

• The existence of an extra Hamiltonian Killing vector field from  $S_{\mathbf{w}}^3$  gives the 2-dimensional Sasaki  $\mathbf{w}$ -cone  $\mathbf{t}_{\mathbf{w}}^+$ .

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$$\Delta_{mv_1,mv_2} = \left(1 - \frac{1}{mv_1}\right)D_1 + \left(1 - \frac{1}{mv_2}\right)D_2$$

consisting of the zero  $D_1$  and infinity  $D_2$  sections of the **projective bundle**  $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$  over N with ramification indices  $mv_1$ ,  $mv_2$ , respectively and n an integer determined by  $l_1$ ,  $l_2$ ,  $\mathbf{w}$ ,  $\mathbf{v}$ .

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• For  $n \neq 0$ , apply the **admissible construction** of Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman on Hamiltonian 2-forms to the ruled Kähler orbifolds  $(S_n, \Delta_{mv_1, mv_2})$ 

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- This gives the Kähler orbifold metric  $g_{(S_n,\Delta)}=\frac{1+r_3}{r}g_{\Sigma_g}+\frac{d_3^2}{\Theta(3)}+\Theta(\mathfrak{z})\theta^2$  where  $\theta$  is a connection 1-form,  $d\theta=n\omega_N$ , 0< r<1,  $\Theta(\mathfrak{z})>0$  and  $-1<\mathfrak{z}<1$ ,  $\Theta(\pm 1)=0$ ,  $\Theta'(-1)=\frac{2}{m\nu_2}$ ,  $\Theta'(1)=-\frac{2}{m\nu_1}$ .

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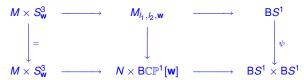
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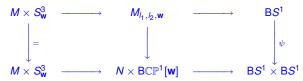
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From join construction get commutative diagram of fibrations:



where BG is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.

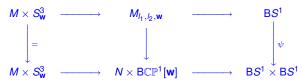
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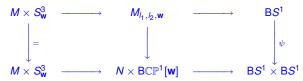
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$$H^r_{orb}(\mathbb{CP}^1[\mathbf{w}], \mathbb{Z}) = H^r(\mathsf{BCP}^1[\mathbf{w}], \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } r = 0, 2, \\ \mathbb{Z}_{w_1 w_2} & \text{for } r > 2 \text{ even,} \\ 0 & \text{for } r \text{ odd.} \end{cases}$$

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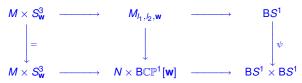
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- Some of the same type of results have been obtained on 5-manifolds whose fundamental group is a non-Abelian extension of  $\pi_1(\Sigma_g)$  in Castañeda's thesis.

THANK YOU!

#### References

#### References

- 1. Extremal Sasakian Metrics on S³-bundles over S², Math. Res. Lett. 18 (2011), no. 01, 181-189.
- 2. Maximal Tori in Contactomorphism Groups, Diff. Geom and Appl. 21, no. 2 (2013), 190-216 (Math arXiv:1003.1903).
- 3. Completely Integrable Contact Hamiltonian Systems and Toric Contact Structures on  $S^2 \times S^3$ , SIGMA Symmetry Integrability Geom. Methods Appl 7 (2011),058, 22.
- 4. (with J. Pati) On the Equivalence Problem for Toric Contact Structures on S<sup>3</sup>-bundles over S<sup>2</sup> Pacific J. Math. 267 (2) (2014), 277-324. (Math arXiv:1204.2209).
- 5. (with C. Tønnesen-Friedman) Extremal Sasakian Geometry on  $T^2 \times S^3$  and Related Manifolds Compositio Math. 149 (2013), 1431-1456 (Math arXiv:1108.2005).
- 6 (with C. Tønnesen-Friedman) Sasakian Manifolds with Perfect Fundamental Groups, African Diaspora Journal of Mathematics 14 (2) (2012), 98-117.
- (with C. Tønnesen-Friedman) Extremal Sasakian Geometry on S<sup>3</sup>-bundles over Riemann Surfaces (to appear in International Mathematical Research Notices) Math ArXiv:1302.0776.
- 8. (with C. Tønnesen-Friedman) The Sasaki Join, Hamiltonian 2-forms, and Sasaki-Einstein Metrics (submitted for publication) Math ArXiv:1309.7067.
- 9. (with C. Tønnesen-Friedman) The Sasaki Join, Hamiltonian 2-forms, and Constant Scalar Curvature (submitted for publication) Math ArXiv:1402.2546.
- 10. (with C. Tønnesen-Friedman) Simply Connected Manifolds with Infinitely many Toric Contact Structures and Constant Scalar Curvature Sasaki Metrics (submitted for publication) Math ArXiv:1404.3999.

General Reference: C.P. B- and K. Galicki, Sasakian Geometry, Oxford University Press, 2008.