The Sasaki Cone
versus
the Kähler Cone

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- **Contact Manifold**(compact)

A **contact 1-form** η *such that*

$$\eta \wedge (d\eta)^n \neq 0.$$

defines a **contact structure**

$$\eta' \sim \eta \iff \eta' = f\eta$$

for some $f \neq 0$, *take* $f > 0$. *or equivalently a codimension 1 sub-bundle* $\mathcal{D} = \mathrm{Ker}\ \eta$ *of* TM.

$(\mathcal{D}, d\eta)$ *symplectic vector bundle*

Unique vector field ξ, *called the* **Reeb vector field**, *satisfying*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

The **characteristic foliation** $\mathcal{F}_\xi$ *each leaf of* $\mathcal{F}_\xi$ *passes through any nbd* U *at most* k *times* $\Longleftrightarrow$ **quasi-regular**, $k = 1 \leftrightarrow$ *regular, otherwise* **irregular**

Quasi-regularity is strong, most contact 1-forms are irregular.

Contact bundle $\mathcal{D} \longrightarrow$ *choose* **almost complex structure** J *extend to* Φ *with* $\Phi\xi = 0$

Get a compatible metric

$$g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$$

Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ *called* **contact metric structure**

The pair $(\mathcal{D}, J)$ *is a strictly pseudo-convex almost CR structure.*

Definition: *The structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *is* **K-contact** *if* $\pounds_\xi g = 0$ *(or* $\pounds_\xi \Phi = 0$*). It is* **Sasakian** *if in addition* $(\mathcal{D}, J)$ *is integrable.*

Note: Quasi-regular $\Rightarrow$ *K-contact.*

- **Manifold** *K-contact* $\iff$ **transverse structure** almost Kähler
- **Manifold Sasakian** $\iff$ **transverse structure** Kähler

$$(\xi, \eta, \Phi, g) \iff (d\eta, J, g_T)$$

Sasaki cone $\kappa(\mathcal{D}, J)$

$= \{\xi\}$ s.t. $\mathcal{S} \in (\mathcal{D}, J)$ is Sasakian $\subset \mathfrak{t}_k$ with $1 \leq k \leq n+1$ $\Rightarrow$ finite dim'l moduli of Sasakian structures with underlying CR structure $(\mathcal{D}, J)$

Alternative Deformation

Fix foliation: $\mathcal{F}_\xi \Rightarrow$ **basic cohomology groups** $H^p_B(\mathcal{F}_\xi)$

Transverse holomorphic $\Rightarrow$ **basic (p,q)-forms** $\Omega^{p,q}_B \Rightarrow$ **basic cohomology groups** $H^{p,q}_B(\mathcal{F}_\xi)$.

Riemannian foliation $\Rightarrow$ **trans-verse Hodge theory** holds.

$\mathcal{S} = (\xi, \eta, \Phi, g)$ Sasakian str.
Space $\mathfrak{F}(\xi)$ of Sasakian structures
$\mathcal{S}' = (\xi, \eta', \Phi', g')$ Reeb v.f. ξ

$\eta \mapsto \eta' = \eta + d\zeta$, ζ is basic. $[d\eta]_B \in H^{1,1}_B(\mathcal{F}_\xi)$ fixed

Subspace $\mathcal{S}(\xi, \bar{J})$ where Φ projects to the complex structure $\bar{J}$ on normal bundle $\nu(\xi) = TM/L_\xi$.

The pair $(\xi, \bar{J})$ defines a **transverse holomorphic structure** on M, and gives infinite dim'l space $\mathcal{S}(\xi, \bar{J})$ of Sasakian structures.

Extremal Sasakian Metrics

(B,Galicki,Simanca)

$$E(g) = \int_M s_g^2 d\mu_g,$$

Varying in $\mathcal{S}(\xi, \bar{J})$ gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic.

Moduli of extremal Sasakian structures $\mathfrak{e}(\mathcal{D}, J) \subset \kappa(\mathcal{D}, J)$

scalar curvature s_g constant $\Rightarrow$ extremal.

Thm (B,Galicki,Simanca) $\mathfrak{e}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.

$\mathfrak{h}(\xi, \bar{J})$ Lie algebra of transversally holomorphic vector fields

Sasaki-Futaki invariant

$$\mathfrak{F}_\xi(X) = \int_M X(\psi_g) d\mu_g$$

where $X \in \mathfrak{h}(\xi, \bar{J})$ and ψ_g is the harmonic rep in $H_B^{1,1}(\mathcal{F}_\xi)$ of the transverse Ricci form ρ_g^T.

If $\mathcal{S}$ is extremal $\mathfrak{F}_\xi(\cdot) = 0 \iff s_g$ is constant.

Assume Sasakian, X, Y sections of $\mathcal{D} \Rightarrow$

$\mathrm{Ric}_g(X,Y) = \mathrm{Ric}_g^T(X,Y) - 2g(X,Y)$

$\rho_g(X,Y) = \rho_g^T(X,Y) - 2d\eta(X,Y)$

$s_g = s_T - 2n$ (scalar curvatures)

s_g constant $\iff s_T$ constant.

ρ_g^T represents $c_1(\mathcal{F}_\xi)$ basic 1st Chern class $\to c_1(\mathcal{D})$

orbifold Boothby-Wang: Manifold M compact with (ξ, η, Φ, g) quasi-regular $\Rightarrow$ quotient $\mathcal{Z} = M/\mathcal{F}_\xi$ almost Kähler orbifold

Converse: $\mathcal{Z} = M/\mathcal{F}_\xi$ almost Kähler orbifold. ω Kähler form with $[\omega] \in H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. Total space M of S^1 orbibundle over $\mathcal{Z}$ has K-contact structure. $(\mathcal{Z}, \omega)$ is projective algebraic orbifold $\iff$ (ξ, η, Φ, g) is Sasakian.

Holomorphy Potentials

$$\mathcal{H}_g^B = \mathrm{Ker}(\bar\partial\partial^\#)^*\bar\partial\partial^\#$$

A map $\partial_g^\# : \mathcal{H}_g^B \to \mathfrak{h}^T(\xi, \bar J)/L_\xi$ gives transversely holomorphic vector fields that are $(1,0)$-gradients.

$\mathcal{S}$ is extremal $\iff$ $s_g \in \mathcal{H}_g^B$.

If Sasaki cone $\kappa(\mathcal{D}, J)$ has dim k $\Rightarrow \mathfrak{Aut}_0(\mathcal{S}) = T^k$ where $\mathcal{S}$ is generic. moment map $\mu : M \to \mathfrak{t}_k^*$ whose image $\subset \mathcal{H}_g^B$. Interested in when $s_g \in \mathrm{Im}\mu$

Extremal: Special Cases

- Sasaki-η-Einstein (SηE)

$\Rightarrow c_1(\mathcal{D}) = 0$

$\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants.

s_g constant. M compact,

$c_1(\mathcal{F}_\xi) \leq 0 \Rightarrow \dim\ \kappa(\mathcal{D}, J) = 1.$

dim 1 $\leftrightarrow$ Transverse homothety

- $\dim\ \kappa(\mathcal{D}, J) > 1. \Rightarrow c_1(\mathcal{F}_\xi) > 0$ or indefinite.

- $c_1(\mathcal{F}_\xi) > 0$ + SηE $\Rightarrow$ SE, many examples including exotic spheres, $k(S^2 \times S^3)$ etc ($\dim\ \kappa(\mathcal{D}, J) = 1.$) (B,Galicki,Nakamaye,Kollár).

• $c_1(\mathcal{F}_\xi) > 0$ and $c_1(\mathcal{D}) = 0$ Toric (dim $\kappa(\mathcal{D}, J) = n + 1$.) $\Rightarrow$ SE (Futaki,Ono,Wang, others)

• $c_1(\mathcal{F}_\xi)$ indefinite $\Rightarrow$ $c_1(\mathcal{D}) \neq 0$ $\Rightarrow$ $\nexists$ SηE. Toric (dim $\kappa(\mathcal{D}, J) = n+1$.) generally not known whether extremal (B,Galicki,Ornea)

Question: When is $\mathfrak{e}(\mathcal{D}, J) = \kappa(\mathcal{D}, J)$?

Only two known cases:

(1) standard CR structure on S^{2n+1}

Toric (dim $\kappa(\mathcal{D}, J) = n + 1$.)

All $\mathcal{S} \in \kappa(\mathcal{D}, J)$ have extremal reps, but only transverse homothety of the round sphere has constant scalar curvature, and only the round sphere is SE. (B, Galicki, Simanca)

(2) The Heisenberg group $\mathfrak{H}^{2n+1}$ with standard CR structure (non-compact), dim $\kappa(\mathcal{D}, J) = n$.
All $\mathcal{S} \in \kappa(\mathcal{D}, J)$ have extremal reps, but there is only one with constanst scalar curvature, SηE with Φ-holomorphic curvature $= -2$

Obstructions to extremality

Lichnerowicz: Lowest eigenvalue of Laplacian

(Gauntlett,Martelli,Sparks,Yau)