# Sasaki join, transverse Hamiltonian 2-forms, and Extremal Sasaki metrics

Special Session on Symplectic and Contact Structures on Manifolds with Special Holonomy

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Joint work with Christina Tønnesen-Friedman.

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- Sasaki bouquets of Sasaki cones can occur when the contactomorphism group has distinct conjugacy classes of maximal tori. For which diffeomorphism types do these occur?

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• The pair  $(\mathfrak{D}, J)$  is a **strictly pseudo-convex almost CR structure** (s $\psi$ CR structure).

#### Definition

The contact metric structure  $\mathcal{S}=(\xi,\eta,\Phi,g)$  is **K-contact** if  $\mathcal{L}_{\xi}g=0$  (or  $\mathcal{L}_{\xi}\Phi=0$ ). It is **Sasakian** if in addition  $(\mathcal{D},J)$  is integrable and the **Transverse Metric**  $g_{\mathcal{D}}$  is Kähler (**Transverse holonomy** U(n)). In the latter case we say that the contact structure  $\mathcal{D}$  is of **Sasaki type**.

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The Sasakian structure  ${\cal S}$  admits the following type of transverse structure:

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- special case: Hamiltonian 2-form of order 1:  $\phi = \frac{-(1+r_3)}{r^2}\omega + \frac{3}{3}d_3 \wedge \theta$ . Here  $\omega$  is the curvature form of a certain line bundle,  $\theta$  is a connection 1-form, r a parameter. (Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman)

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- We call this an Admissible Transverse Structure. This is compatible with our next construction.

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- $M_1 \star_{l_1, l_2} M_2$  has a natural quasi-regular Sasakian structure  $\mathcal{S}_{l_1, l_2}$  for all relatively prime positive integers  $l_1, l_2$ . Fixing  $l_1, l_2$  fixes the contact orbifold. It is a smooth manifold iff  $\gcd(v_1 l_2, v_2 l_1) = 1$  where  $v_i$  is the order of orbifold  $\mathcal{Z}_i$ .

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- The transverse holonomy of  $M \star_{l_1, l_2} S_{\mathbf{w}}^3$  is generically  $U(n_1) \times U(2)$ .

## The Join Construction

- Join Construction: Given quasi-regular Sasakian manifolds  $\pi_i: M_i \longrightarrow \mathcal{Z}_i$  with Dim  $M_i = n_i$  for i = 1, 2.
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- $M_1 \star_{l_1, l_2} M_2$  has a natural quasi-regular Sasakian structure  $\mathcal{S}_{l_1, l_2}$  for all relatively prime positive integers  $l_1, l_2$ . Fixing  $l_1, l_2$  fixes the contact orbifold. It is a smooth manifold iff  $\gcd(v_1 l_2, v_2 l_1) = 1$  where  $v_i$  is the order of orbifold  $\mathcal{Z}_i$ .
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- Take  $\pi_2: M_2 \longrightarrow \mathcal{Z}_2$  to be the  $S^1$  orbibundle  $\pi_2: S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1[\mathbf{w}]$  determined by a weighted  $S^1$  action on  $S^3$  with weights  $\mathbf{w} = (w_1, w_2)$  satisfying  $\gcd(I_2, I_1 w_1) = 1$ , and  $M_1 = M$  regular Sasaki manifold whose quotient is a compact Kähler manifold N.
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### Extremal Sasakian metrics (B-Galicki-Simanca)

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- The Sasaki-Futaki invariant  $\mathfrak{F}(X)=\int_M X(\psi_g)d\mu_g$  where X is transversely holomorphic and  $\psi_g$  is the Ricci potential satisfying  $\rho^T=\rho_h^T+i\partial\bar\partial\psi_g$  where  $\rho^T$  is the transverse Ricci form and  $\rho_h^T$  is its harmonic part. An extremal Sasaki metric g has constant scalar curvature if and only if  $\mathfrak{F}=0$ .

## The Extremal Set

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- If  $0 < g \le 4$  all 2-dimensional Sasaki cones on  $S^3$ -bundles over  $\Sigma_g$  obtained by our construction have  $\mathfrak{e}(\mathfrak{D},J) = \kappa(\mathfrak{D},J)$  (B-,Tønnesen-Friedman).

• Recall  $M \star_{l_1, l_2} S_{\mathbf{w}}^3$  is a  $L(l_2; l_1 w_1, l_1, w_2)$  bundle over a Kähler manifold N.

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# Theorem (B-,Tønnesen-Friedman)

Let  $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S_{\mathbf{w}}^3$  be the  $S_{\mathbf{w}}^3$ -join with a regular Sasaki manifold M which is an  $S^1$ -bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector  $\mathbf{w} = (w_1,w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$  with relatively prime components satisfying  $w_1 > w_2$  there exists a Reeb vector field  $\xi_{\mathbf{v}}$  in a 2-dimensional sub cone, the  $\mathbf{w}$ -cone, of the Sasaki cone on  $M_{l_1,l_2,\mathbf{w}}$  such that the corresponding ray of Sasakian structures  $S_a = (a^{-1}\xi_{\mathbf{v}}, a_{N}, \Phi, g_a)$  has constant scalar curvature. Moreover, given the cohomology ring of M, the cohomology ring of  $M_{l_1,l_2,\mathbf{w}}$  can be determined.

- Remark: Most of the CSC Sasakian structures are irregular.
- When N is positive **KE** get **SE** metric on  $M_{l_1,l_2,\mathbf{w}}$  for appropriate choice of  $(l_1,l_2)$ .
- In special cases determine the diffeomorphism (homeomorphism, homotopy) types.
- Two Cases: Reason: Interference in  $E_2$  term of a Leray-Serre spectral sequence.

  - $2 \dim_{\mathbb{C}} N > 1.$

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- Some of the same type of results can be obtained on 5-manifolds whose fundamental group is a non-Abelian extension of  $\pi_1(\Sigma_q)$ .

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- Take N to be a positive KE manifold, and let

$$I_1 = \frac{J_N}{\gcd(w_1 + w_2, J_N)}, \qquad I_2 = \frac{w_1 + w_2}{\gcd(w_1 + w_2, J_N)}$$

where  $\mathfrak{I}_N$  denotes the Fano index of N. Then for each vector  $\mathbf{w}=(w_1,w_2)\in\mathbb{Z}^+\times\mathbb{Z}^+$  with relatively prime components satisfying  $w_1>w_2$  there exists a Reeb vector field  $\xi_{\mathbf{v}}$  in the 2-dimensional **w**-Sasaki cone on  $M_{l_1,l_2,\mathbf{w}}$  such that the corresponding Sasakian structure  $\mathcal{S}=(\xi_{\mathbf{v}},\eta_{\mathbf{v}},\Phi,g)$  is **Sasaki-Einstein**.

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• In this case the Sasakian structure associated to every single ray,  $\xi_{\mathbf{V}}$ , in the **w**-Sasaki cone is a **Sasaki-Ricci soliton** and  $\mathfrak{e}(\mathcal{D}, J) = \kappa(\mathcal{D}, J)$ .

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- Special Case:  $N = \mathbb{CP}^p$  with Fubini-Study metric and p > 1, so  $M = S^{2p+1}$  with its standard Sasaki-Einstein metric. Here  $l_1, l_2$  are arbitrary relatively prime positive integers.

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- In this case the **cohomology ring** of the join  $M_{l_1,l_2,\mathbf{w}}^{2p+1} = S^{2p+1} \star_{l_1,l_2} S^3_{\mathbf{w}}$  is

$$\mathbb{Z}[x,y]/(w_1w_2l_1^2x^2,x^{p+1},x^2y,y^2)$$

where x, y are classes of degree 2 and 2p + 1, respectively.

• Specialize further: Take p=2. Then  $M_{l'_1,l'_2,\mathbf{w}'}^7$  and  $M_{l_1,l_2,\mathbf{w}}^7$  are homotopy equivalent if and only if

$$\begin{aligned} w_1'w_2'(l_1')^2 &= w_1w_2l_1^2, \qquad (l_1')^2(w_1' + w_2')^2 - l_1^2(w_1 + w_2)^2 \equiv 0 \mod 3 \in \mathbb{Z}_{w_1w_2l_1^2} \\ \\ l_2' &\equiv l_2 \mod 2, , \qquad (l_2')^3 \pm l_2^3 \equiv 0 \mod w_1w_2l_1^2. \end{aligned}$$

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• Example: The eight 7-manifolds with  $(l_1, l_2, w_1, w_2) =$ 

$$(1, 2182, 6545, 1), (1, 438, 1309, 5), (1, 202, 935, 7), (1, 134, 385, 17),$$

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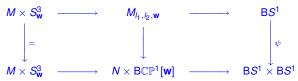
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   Generally they are fairly difficult to compute.
- Example: Simple case;  $M^7$  is homogeneous, i.e.  $\mathbf{w} = (1,1)$ . Take  $l_1 = 5$ . Then  $M^7_{5,\frac{l}{2},(1,1)}$  and  $M^7_{5,\frac{l}{2},(1,1)}$  are homeomorphic if and only if  $l_2' \equiv l_2 \mod 50$ , and they are diffeomorphic if and only if  $l_2' \equiv l_2 \mod 100$ . There is a countable infinity of contact structures of Sasaki type on each diffeomorphism type. Furthermore, they all admit **CSC** Sasaki metrics.

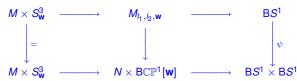
# Outline of proof of Fundamental Theorem:

From join construction get commutative diagram of fibrations:



where BG is the classifying space of a group G or Haefliger's classifying space of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations.

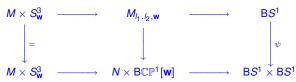
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• Given the differentials in the **spectral sequence** of the fibration  $M \longrightarrow N \longrightarrow BS^1$ , one can use the commutative diagram to compute the **cohomology ring** of the **contact** manifold  $M_{l_1,l_2,\mathbf{w}}$ .

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  N

  BS

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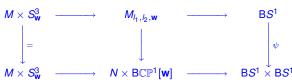
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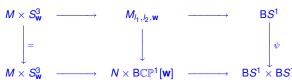
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- The existence of an extra Hamiltonian Killing vector field gives 2-dimensional Sasaki cone  $\kappa(\mathcal{D}_{l_1,l_2,\mathbf{w}},J)$ .

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- The existence of an extra Hamiltonian Killing vector field gives 2-dimensional Sasaki cone  $\kappa(\mathcal{D}_{l_1,l_2,\mathbf{w}},J)$ .
- The quotient space of the  $S^1$ -action generated by any quasi-regular Reeb vector field  $\xi_{\mathbf{V}} \in \kappa(\mathcal{D}_{l_1,l_2,\mathbf{w}},J)$  is a ruled orbifold  $(S_n,\Delta_{mv_1,mv_2})$  with a branch divisor  $\Delta_{mv_1,mv_2}$  consisting of the zero and infinity sections of the **projective bundle**  $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$  over N with ramification indices  $mv_1, mv_2$ , respectively and n an integer determined by  $l_1, l_2, \mathbf{w}, \mathbf{v}$ .

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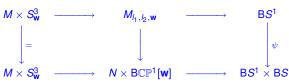
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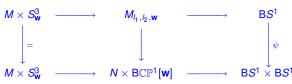
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- There exist a countably infinite number of aspherical contact 5-manifolds with perfect fundamental group and the integral cohomology ring of  $S^2 \times S^3$  that admit CSC Sasaki metrics. Moreover, there are such manifolds that admit a ray of Sasaki- $\eta$ -Einstein metrics (hence, Lorentzian Sasaki-Einstein metrics).

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