Sasaki join, transverse Hamiltonian 2-forms, and Extremal Sasaki metrics

Special Session on Symplectic and Contact Structures on Manifolds with Special Holonomy

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University of New Mexico

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PROGRAM: FOCAL POINTS

- Joint work with Christina Tønnesen-Friedman.

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- **Sasaki bouquets** of Sasaki cones can occur when the contactomorphism group has distinct conjugacy classes of maximal tori. For which **diffeomorphism types** do these occur?

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- The pair $(\mathcal{D}, J)$ is a **strictly pseudo-convex almost CR structure** (sψCR structure).

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The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **Transverse Metric** $g_{\mathcal{D}}$ is Kähler (**Transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

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- We call this an **Admissible Transverse Structure**. This is compatible with our next construction.

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- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure $\mathcal{S}_{l_1,l_2}$ for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\upsilon_1 l_2, \upsilon_2 l_1) = 1$ where υ_i is the order of orbifold $\mathcal{Z}_i$.
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- The Sasaki-Futaki invariant $\mathfrak{F}(X) = \int_M X(\psi_g) d\mu_g$ where X is transversely holomorphic and ψ_g is the Ricci potential satisfying $\rho^T = \rho_h^T + i\partial\bar\partial\psi_g$ where ρ^T is the transverse Ricci form and ρ_h^T is its harmonic part. An extremal Sasaki metric g has constant scalar curvature if and only if $\mathfrak{F} = 0$.

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- A noncompact example with $\mathfrak{e}(\mathcal{D}, J) = \kappa(\mathcal{D}, J)$ is the Heisenberg group.
- If $0 < g \leq 4$ all 2-dimensional Sasaki cones on S^3-bundles over Σ_g obtained by our construction have $\mathfrak{e}(\mathcal{D}, J) = \kappa(\mathcal{D}, J)$ (B-,Tønnesen-Friedman).

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- Some of the same type of results can be obtained on 5-manifolds whose fundamental group is a non-Abelian extension of $\pi_1(\Sigma_g)$.

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- In this case the **cohomology ring** of the join $M^{2p+1}_{l_1,l_2,\mathbf{w}} = S^{2p+1} \star_{l_1,l_2} S^3_{\mathbf{w}}$ is

$$\mathbb{Z}[x, y]/(w_1 w_2 l_1^2 x^2, x^{p+1}, x^2 y, y^2)$$

  where x, y are classes of degree 2 and $2p+1$, respectively.

Seven Dimensional Case:

- Specialize further: Take $p=2$. Then $M^7_{l_1',l_2',\mathbf{w}'}$ and $M^7_{l_1,l_2,\mathbf{w}}$ are homotopy equivalent if and only if

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- **Example**: Simple case; M^7 is homogeneous, i.e. $\mathbf{w} = (1, 1)$. Take $l_1 = 5$. Then $M^7_{5, l_2', (1,1)}$ and $M^7_{5, l_2, (1,1)}$ are homeomorphic if and only if $l_2' \equiv l_2 \mod 50$, and they are diffeomorphic if and only if $l_2' \equiv l_2 \mod 100$. There is a countable infinity of contact structures of Sasaki type on each diffeomorphism type. Furthermore, they all admit **CSC** Sasaki metrics.

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- From join construction get commutative diagram of fibrations:

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