A Panorama of Sasakian Geometry

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WORKSHOP ON GRADED ALGEBRA, GEOMETRY AND RELATED TOPICS
Mérida, México

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7. Most contact forms in a contact structure $\mathcal{D}$ are **irregular**
8. We can choose a **compatible almost complex structure** J on $\mathcal{D}$, that is one that satisfies the two conditions
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9. The **almost complex structure** J extends to an endomorphism Φ of TM satisfying $\Phi\xi = 0$.

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes 1\!\!1) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.

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  3. If $\mathcal{S}$ is irregular, then the closure $\bar{\mathcal{F}}_\xi$ is a **torus** T^k of dimension $1 \leq k \leq n+1$.

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Definition

The contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable and the **transverse metric** $g_{\mathcal{D}}$ is Kähler (**transverse holonomy** $U(n)$). In the latter case we say that the contact structure $\mathcal{D}$ is of **Sasaki type**.

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Sasakian Geometry, continued

- There is a 'canonical' compatible metric $g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$. Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**. Contact metric manifold $(M, \mathcal{S})$.
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- If ψ is a **Killing spinor** then $V^\psi = \sum_i g(\psi, E_i \cdot \psi) E_i$ is a **Killing vector field**.

Killing Spinors and Sasaki-Einstein Manifolds

- A Riemannian **spin manifold** (M^n, g) is of **type** (p, q) if it carries exactly p linearly independent real **Killing spinors** with $\alpha > 0$ and exactly q linearly independent real **Killing spinors** with $\alpha < 0$.

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- The only other compact manifolds that admit **real Killing spinors** are the **round spheres** of any dimension and manifolds of dimension 6 and 7 whose **Riemannian cone** has holonomy G_2 and $\mathrm{Spin}7$, respectively.

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- We denote by $\mathbb{R}^{n|m}$ the space $\mathbb{R}^n$ endowed with the **structure sheaf** $\mathcal{A} = \mathcal{C}^\infty[\theta^1, \cdots, \theta^m]$ where the θ^i generate an exterior algebra $\mathcal{A}^1$ and $\mathcal{C}^\infty$ is the sheaf of smooth functions on $\mathbb{R}^n$.

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- This generates an **action** of a **Lie supergroup** $\mathfrak{G}$ on $\mathbb{R}^{1|1}$, namely $(t, \theta) \mapsto (t + t' + \theta'\theta, \theta + \theta')$.

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- This can be described by a map $\Phi : \mathbb{R}^{1|1} \longrightarrow N$ such that $x = \iota^* \Phi$ and $\varphi = \iota^* D\Phi$ where $D = \partial_\theta - \theta\partial_t$ and $\iota : \mathbb{R} \longrightarrow \mathbb{R}^{1|1}$ is the inclusion defined by $\iota^* t = t$ and $\iota^* \theta = 0$.

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- $\mathcal{L}$ is invariant under the **Lie supergroup** $\mathfrak{G}$ generated by $\partial_t, Q = \partial_\theta + \theta\partial_t$, that is $\mathcal{L}$ is **supersymmetric**.

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- Given a **Sasaki-Einstein manifold** $|M|$, can we associate a **supermanifold structure** $(M, \mathcal{A})$ on $|M|$ such that the Killing spinors "generate" a Lie supergroup action on $(M, \mathcal{A})$?

MUCHAS GRACIAS POR SU ATENCIÓN (THANK YOU FOR YOUR ATTENTION)!!

MUCHAS FELICIDADES, ADOLFO