L^p norms and global harmonic analysis

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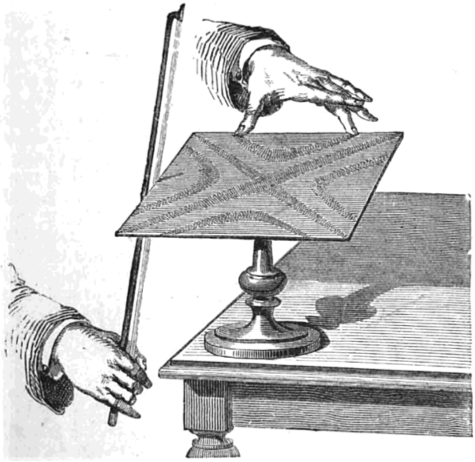
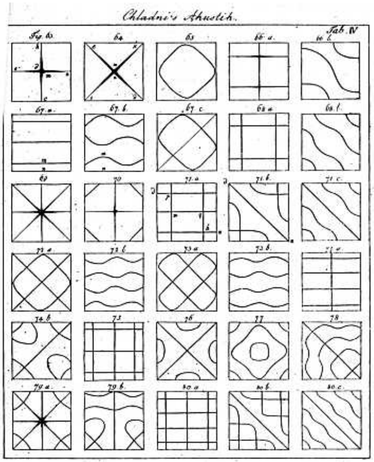
Chladni's experiment (circa 1787)



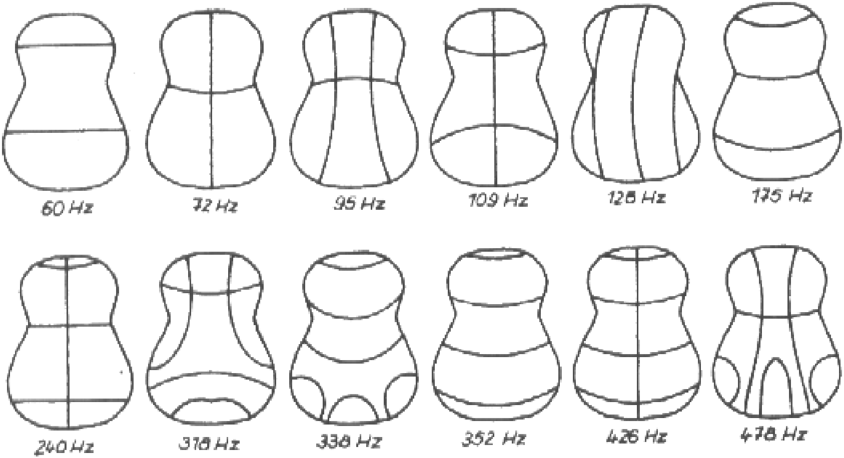
Bowing a dusted plate

Chladni figures on a square plate



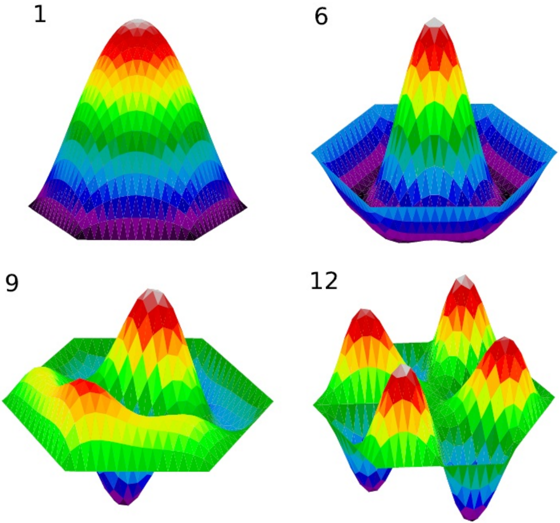
From the book *Die Akustik* by Ernst Chladni

Changing the shape of the plate



Uploaded to Wikipedia by Denis Diderot

Amplitudes of vibrational modes



Helmholtz equation

- Led to study the Helmholtz equation on a domain

$$(\lambda^2 + \Delta)\varphi_\lambda(x) = 0$$

- Every φ_λ gives rise to solutions to the wave equation

$$(-\partial_t^2 + \Delta)(e^{\pm it\lambda}\varphi_\lambda(x)) = 0$$

- Fermat's principle suggests the mass of φ_λ should in some sense be invariant under the billiard dynamics (paths of least action) on the plate as $\lambda \to \infty$

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L^p norms

- For a compact domain M and $f : M \to \mathbb{C}$ continuous, define

$$\|f\|_{L^p(M)} := \left(\int_M |f(x)|^p \, dx\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$
$$\|f\|_{L^\infty(M)} := \max_{x \in M} |f(x)|,$$

- Helmholtz sol'ns often normalized to have norm 1 in $L^2(M)$
- For $2 < p < \infty$, L^p norms will be sensitive to both the extrema of the function and the regions where the extrema is nearly obtained

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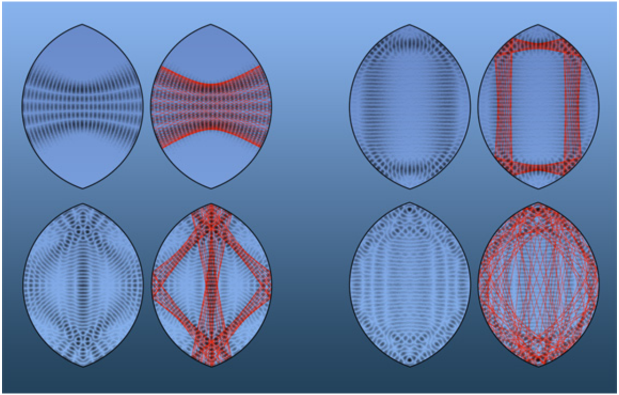
Scarring



By Eric Heller

Laplacian on a compact manifold

- Let (M, g) be a C^∞, *boundaryless*, compact Riemannian manifold, $\dim(M) = n \geq 2$
- Let Δ_g be the Laplace-Beltrami operator, in coordinates

$$\Delta_g u = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g_{kl}}} \partial_i \left(g^{ij} \sqrt{\det g_{kl}}\, \partial_j u \right)$$

- Self-adjoint w.r.t. Riemannian measure dV_g

$$\int_M \Delta_g u\, \overline{v}\, dV_g = \int_M u\, \overline{\Delta_g v}\, dV_g,$$
$$\int_M \Delta_g u\, \overline{u}\, dV_g = -\int_M |\nabla_g u|^2 dV_g$$

Eigenfunctions on a compact manifold

- Compactness of $M \Longrightarrow$ spectrum of Δ_g is discrete: there exists a sequence $\{\varphi_{\lambda_j}\}_{j=1}^{\infty}$ of eigenfunctions forming an O.N. basis for $L^2(M)$

$$\Delta_g \varphi_{\lambda_j} = -\lambda_j^2 \varphi_{\lambda_j}, \qquad \sqrt{-\Delta_g}\varphi_{\lambda_j} = \lambda_j \varphi_{\lambda_j}$$

- Eigenspaces are finite dimensional, but may have high multiplicity (perhaps even growing with λ)
- Notation: φ_λ is any admissible solution

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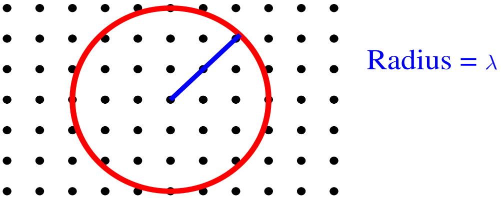
Example: Flat torus

- Let $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$, endowed with the flat metric (a.k.a. $[-\pi, \pi]^n$ with opposite sides "glued" together)
- Laplacian is the familiar $\Delta u = \sum_1^n \frac{\partial^2 u}{\partial x_j^2}$
- Eigenfunctions are linear combos of Fourier modes

$$\varphi_\lambda(x) = \frac{1}{(2\pi)^n} \sum_{|m|=\lambda} a_m e^{ix\cdot m}, \qquad \sum_{|m|=\lambda} |a_m|^2 = 1 \qquad (m \in \mathbb{Z}^n)$$



Example: $\mathbb{S}^n$, canonical sphere in $\mathbb{R}^{n+1}$

- On $\mathbb{S}^2$, with spherical coordinates (θ, ϕ)

$$\Delta_g u = \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta u\right) + \frac{1}{\sin^2\theta}\partial_\phi^2 u$$

- E'functions are linear combos of $Y_l^m(\theta,\phi) = e^{im\phi}P_l^m(\cos\theta)$ (P_l^m an associated Legendre polynomial)
- For $n \geq 2$, spectrum of Δ_g on $\mathbb{S}^n$ is:

$$\mathrm{spec}(\sqrt{-\Delta_g}) = \left\{\sqrt{k(k+n-1)} = k + \mathcal{O}(k^{-1}) : k = 0, 1, 2, \dots\right\}$$

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L^p bounds on eigenfunctions

- Main problem: given (M, g) and the unit sphere in an eigenspace

$$V_\lambda = \{\varphi_\lambda : \Delta_g \varphi_\lambda = -\lambda^2 \varphi_\lambda, \|\varphi_\lambda\|_{L^2} = 1\}$$

find (best possible) upper bounds on

$$\sup_{\varphi_\lambda \in V_\lambda} \|\varphi_\lambda\|_{L^p(M)}, \qquad 2 < p \leq \infty$$

- Might expect the following type, for some $\delta(p) > 0$

$$\|\varphi_\lambda\|_{L^p(M)} \leq C\lambda^{\delta(p)} \qquad \text{for } \|\varphi_\lambda\|_{L^2(M)} = 1$$

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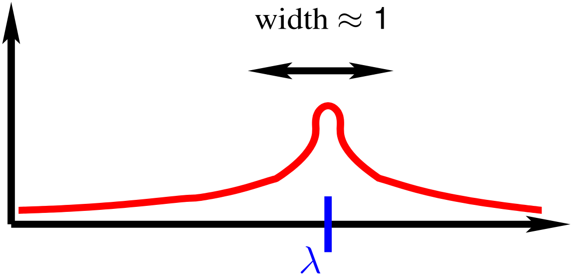
Wave approach to eigenfunctions (see Sogge, *Fourier Integrals in Classical Analysis*)

- Let $\chi \in \mathcal{S}(\mathbb{R})$, satisfy $\text{supp}(\widehat{\chi}) \subset (-\epsilon, \epsilon)$, $\chi(0) = 1$, define

$$\chi(\lambda - \sqrt{-\Delta_g})f = \sum_{j=1}^{\infty} \chi(\lambda - \lambda_j)\langle f, \varphi_{\lambda_j}\rangle\, \varphi_{\lambda_j}$$

so that $\chi(\lambda - \sqrt{-\Delta_g})\varphi_\lambda = \varphi_\lambda$



Wave approach to eigenfunctions

- Formally, using Fourier integrals

$$\chi(\lambda - \sqrt{-\Delta_g}) = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} e^{it\lambda} e^{-it\sqrt{-\Delta_g}} \widehat{\chi}(t)\, dt$$

- $e^{-it\sqrt{-\Delta_g}}$ is the half-wave evolution, a solution map for the wave eqn, i.e.

$$e^{-it\sqrt{-\Delta_g}} f = \sum_{j=1}^{\infty} e^{-it\lambda_j} \langle f, \varphi_{\lambda_j} \rangle \varphi_{\lambda_j}$$

- Lax parametrix + stationary phase yields

$$\chi(\lambda - \sqrt{-\Delta_g}) v(x) = \int_M e^{i\lambda d_g(x,y)} a_\lambda(x,y) v(y)\, dy + \text{ small error}$$

$$\operatorname{supp}(a_\lambda) \subset \{(x,y) \in M \times M : d_g(x,y) \leq \epsilon\}$$

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Limitations to the wave approach

- The dream: replace $\chi(\lambda - \sqrt{-\Delta_g})$ by $\tilde{\chi}_\lambda(\sqrt{-\Delta_g})$ where

$$\text{supp}(\tilde{\chi}_\lambda) \cap \text{spec}(\sqrt{-\Delta_g}) = \{\lambda\}$$

$$\tilde{\chi}_\lambda(\sqrt{-\Delta_g}) = \frac{1}{2\pi} \int e^{it\lambda} e^{-it\sqrt{-\Delta_g}} \widehat{\tilde{\chi}_\lambda}(t)\, dt$$

- Fourier transform of $\tilde{\chi}_\lambda$ loses the compact support, so must understand the wave kernel for all $t \in \mathbb{R}$
- Weyl's law and its consequences:

$$\#\{\lambda_j : \lambda_j \leq \lambda\} = c_n \lambda^n + \mathcal{O}(\lambda^{n-1})$$

$$\#\{\lambda_j : \lambda_j \in [\lambda, \lambda + 1]\} = \mathcal{O}(\lambda^{n-1})$$

  so *lots* of eigenfunctions in the range of $\chi(\lambda - \sqrt{-\Delta_g})$, typically a *crude* approximation to eigenspace projection

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L^p bounds on spectral clusters

Theorem: "Universal" L^p bounds (Sogge '88)

If $\dim(M) = n \geq 2$ and $g \in C^\infty$, then

$$\|\chi(\lambda - \sqrt{-\Delta_g})f\|_{L^p(M)} \leq C\lambda^{\delta(p)}\|f\|_{L^2(M)}$$

$$\delta(p) = \begin{cases} \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq \frac{2(n+1)}{n-1} \\ \frac{n-1}{2} - \frac{n}{p}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \end{cases}$$

Consequently, $\|\varphi_\lambda\|_{L^p(M)} = \mathcal{O}(\lambda^{\delta(p)})$ as $\lambda \to \infty$

- At best, growth rate $\delta(p)$ is sharp for the operator $\chi(\lambda - \sqrt{-\Delta_g})$, but not necessarily eigenspace projections
- $\mathbb{S}^n$ is exceptional: $\text{spec}(\sqrt{-\Delta_g}) = \{k + \mathcal{O}(1/k) : k = 0, 1, 2, \dots\}$

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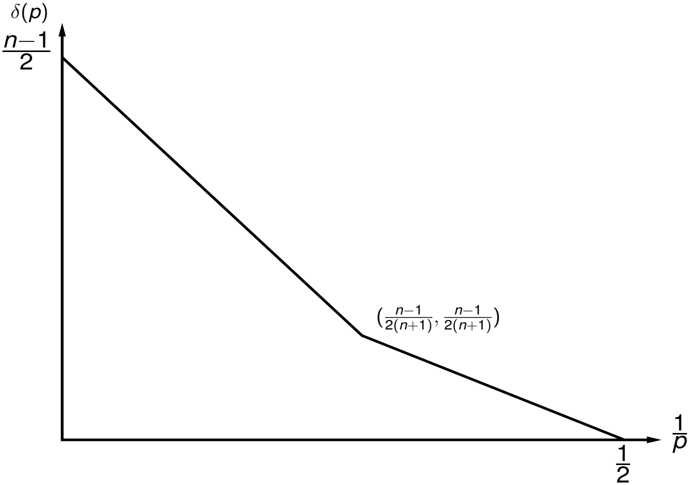

$\delta(p)$
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$\frac{1}{p}$
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Zonal harmonics on $\mathbb{S}^n$

- Sogge: On $\mathbb{S}^n$, can find φ_λ satisfying for enough x

$$|\varphi_\lambda(x)| \approx \lambda^{\frac{n-1}{2}}(1+\lambda d_g(x,x_0))^{-\frac{n-1}{2}} \quad \text{near } x_0 \in M,$$

hence the $\delta(p) = \frac{n-1}{2} - \frac{n}{p}$ cannot be improved for $\frac{2(n+1)}{n-1} \leq p \leq \infty$

$$\|\varphi_\lambda\|_{L^p(M)} \approx \lambda^{\frac{n-1}{2}-\frac{n}{p}}$$

- On $\mathbb{S}^n$, these are zonal harmonics (exact eigenfunctions)

Highest weight/sectoral harmonics on $\mathbb{S}^n$

- φ_λ concentrated in a $\lambda^{-\frac{1}{2}}$-nbd of a closed geodesic γ

$$\mathcal{T}_{\lambda^{-1/2}}(\gamma) = \{x : d_g(x,\gamma) \lesssim \lambda^{-\frac{1}{2}}\},$$

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$$\|\varphi_\lambda\|_{L^p(M)} \approx \left[\mathrm{Vol}(\mathcal{T}_{\lambda^{-1/2}}(\gamma))\right]^{\frac{1}{p}-\frac{1}{2}} \approx \lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})}$$

- On $\mathbb{S}^n$, these are the *highest weight harmonics*
- Underscores that $p = \frac{2(n+1)}{n-1}$ is the *critical* exponent. *Both* mass concentration profiles saturate the growth rate.
- Pietromonaco honors thesis: worked out Sogge's examples in detail

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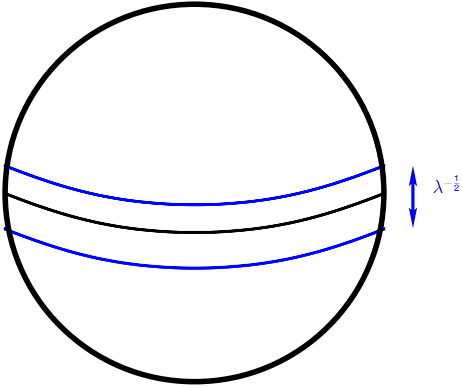
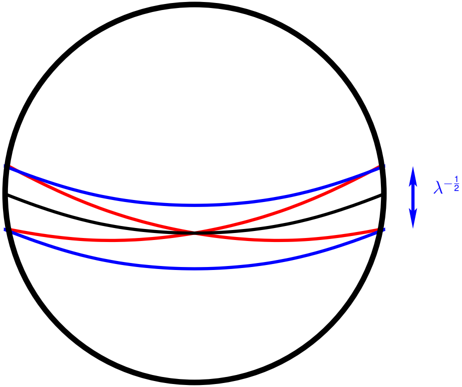
Highest weight harmonics on $\mathbb{S}^n$

- Mass of φ_λ concentrated in a $\lambda^{-\frac{1}{2}}$ neighborhood of the equator



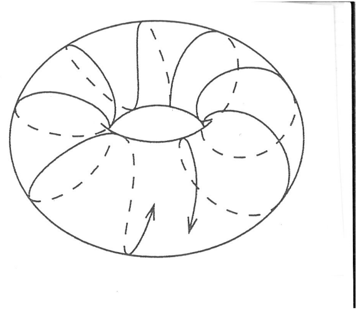
Highest weight harmonics on $\mathbb{S}^n$

- Equator is a stable, degenerate orbit



Flat and negatively curved manifolds

- No such stable orbits on the torus: e.g. image of lines with irrational slope under the quotient map are dense



(Image from J. Lee, *Introduction to Smooth Manifolds*)

- If M has negative sectional curvatures, the geodesic flow is unstable

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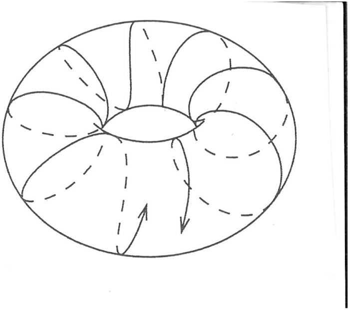

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$\mathbb{S}^n$ is *too* special

Main question for general (M, g): When can the $\mathcal{O}(\lambda^{\delta(p)})$ bounds be improved?

- Zygmund: $\|\varphi_\lambda\|_{L^4(\mathbb{T}^2)} \le C$
- Classical number theory: $\|\varphi_\lambda\|_{L^\infty(\mathbb{T}^2)} \le C_\epsilon \lambda^\epsilon$, any $\epsilon > 0$
- Bourgain, Bourgain-Demeter: improvements on $\delta(p)$ exponent for flat tori
- Iwaniec-Sarnak: power improvements for certain (noncompact) arithmetic quotients of the hyperbolic plane
- Sogge-Zelditch: maximal L^∞ growth occurs only when “full measure" family of geodesics issuing from a point loop back at a common time

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Nonpositive curvature

- Negative curvature: geodesic flow behaves chaotically, stable and unstable manifolds are invariant under geodesic flow (Anosov)
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Wave approach for nonpositive curvature

- As before $\chi \in \mathcal{S}(\mathbb{R})$ with $\widehat{\chi} \in C^\infty$ and compactly supported in $(-\epsilon, \epsilon)$, $\chi(0) = 1$
- This time consider with $T = T(\lambda) = \log \lambda$

$$\chi(T(\lambda - \sqrt{-\Delta_g})) = \frac{1}{2\pi T} \int_{-\epsilon T}^{\epsilon T} e^{it\lambda} e^{-it\sqrt{-\Delta_g}} \widehat{\chi}\left(\frac{t}{T}\right) dt$$

- Yields a gain of $\frac{1}{T}$, but have to analyze $e^{-it\sqrt{-\Delta_g}}$ over large time scales
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Ehrenfest time

- Coherent states/wave packets at frequency λ, consider Gaussian concentrated at scales Δx in space, $\Delta \xi$ in frequency
- Take localization scales $\Delta x \approx \lambda^{-1/2}$, $\Delta \xi \approx \lambda^{1/2}$, finest possible while respecting uncertainty principle and (approximate) invariance under bicharacteristic flow
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Improvements beneath the critical exponent

Thm (B.-Sogge '15, to app. *Comm. Math. Phys.*, *J. Diff. Geom.*)

Suppose (M, g) has nonpositive sectional curvatures. Then for any $2 < p < \frac{2(n+1)}{n-1}$, there exists an exponent $\sigma = \sigma(p, n) > 0$ such that

$$\|\varphi_\lambda\|_{L^p(M)} = \mathcal{O}\left(\frac{\lambda^{\delta(p)}}{(\log \lambda)^\sigma}\right)$$

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- Earlier work: Sogge, Sogge-Zelditch, B.-Sogge
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Restrictions to geodesic segments

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Suppose (M, g) has nonpositive sectional curvatures and $n = 2$. Then for any unit length geodesic segment γ

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