ON MULTILINEAR SPECTRAL CLUSTER ESTIMATES FOR MANIFOLDS WITH BOUNDARY

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1. Introduction

Let (M^n, g) be a smooth, compact n-dimensional Riemannian manifold with boundary and let Δ be the corresponding Laplace-Beltrami operator acting on functions. If the boundary is non-empty, we assume that either Dirichlet or Neumann conditions are imposed along ∂M^n.

Consider the operators χ_λ defined as projection onto the subspace spanned by the Dirichlet (or Neumann) eigenfunctions whose corresponding eigenvalues $-\lambda_j^2$ satisfy $\lambda_j \in [\lambda - 1, \lambda]$. In the case that ∂M^n is empty, it was established in [10] that the following, best possible $L^2 \to L^q$ estimates hold for χ_λ:

$$\|\chi_\lambda\|_{L^2 \to L^q} \lesssim \begin{cases} \lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})} & 2 \le q \le \frac{2(n+1)}{n-1} \\ \lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} & \frac{2(n+1)}{n-1} \le q \le \infty \end{cases} \tag{1.1}$$

Recently, in [1] and [2], Burq, Gérard, and Tzvetkov established multilinear versions of these estimates, also under the assumption that the boundary of M is empty. To state these, suppose that $\lambda \ge \mu \ge \nu \ge 1$. Then

$$\|\chi_\lambda f\, \chi_\mu g\|_{L^2(M)} \lesssim \Lambda(\mu) \|f\|_{L^2(M)} \|g\|_{L^2(M)} \tag{1.2}$$

$$\|\chi_\lambda f\, \chi_\mu g\, \chi_\nu h\|_{L^2(M)} \lesssim (\mu\nu)^{\frac{2n-3}{4}} \|f\|_{L^2(M)} \|g\|_{L^2(M)} \|h\|_{L^2(M)} \tag{1.3}$$

where in the first estimate

$$\Lambda(\mu) = \begin{cases} \mu^{\frac{1}{4}} & n = 2 \\ \mu^{\frac{1}{2}} (\log \mu)^{\frac{1}{2}} & n = 3 \\ \mu^{\frac{n-2}{2}} & n \ge 4 \end{cases}$$

With the exception of the logarithmic loss for $n = 3$, the linear estimate (1.1) with $q = 4$ follows as a corollary of the bilinear estimate (1.2), by taking $\lambda = \mu$ and $f = g$. Similarly, the trilinear estimate (1.3) implies (1.1) with $q = 6$. Moreover, by taking h constant and $\nu = 1$, (1.3) implies (1.2) in case $n = 2$. For $n \ge 4$, however, the trilinear estimate can be improved by using (1.2) together with the L^∞ bounds (1.1) on h.

In the case where ∂M^n is nonempty, the issue of spectral cluster estimates is considerably more intricate. Here the Rayleigh whispering gallery modes provide examples of spectral clusters which concentrate in a $\lambda^{-\frac{2}{3}} \times \lambda^{-\frac{n-2}{2}}$ neighborhood

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of a boundary geodesic (see Grieser [4]). These examples show that one cannot achieve linear spectral cluster estimates better than

$$\|\chi_\lambda\|_{L^2\to L^q} \lesssim \begin{cases} \lambda^{(\frac{2}{3}+\frac{n-2}{2})(\frac{1}{2}-\frac{1}{q})} & 2\le q\le \frac{6n+4}{3n-4} \\ \lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} & \frac{6n+4}{3n-4}\le q\le\infty \end{cases} \tag{1.4}$$

The estimates (1.4) were recently proven for dimension $n=2$ in [9], along with partial results in higher dimensions. The question of whether or not they hold in general in higher dimensions remains open.

In this work, we establish the following multilinear spectral cluster estimates on a general Riemannian manifold with boundary. We restrict attention to dimensions $n=2,3$, since that is where our results are in some context sharp.

Theorem 1.1. *Let (M^n, g) and χ_λ be as above, with either Dirichlet or Neumann eigenfunctions, and let $\lambda\ge\mu\ge\nu$. Then the following bilinear estimate holds*

$$\|\chi_\lambda f\,\chi_\mu g\|_{L^2(M)} \lesssim \Lambda(\mu)\,\|f\|_{L^2(M)}\|g\|_{L^2(M)} \tag{1.5}$$

with Λ defined as

$$\Lambda(\mu)=\begin{cases}\mu^{\frac{1}{3}} & n=2\\ \mu^{\frac{2}{3}}(\log\mu)^{\frac{1}{2}} & n=3\end{cases}$$

In addition, the following trilinear estimate holds for $n=2,3$

$$\|\chi_\lambda f\,\chi_\mu g\,\chi_\nu h\|_{L^2(M)} \lesssim (\mu\nu)^{\frac{3n-4}{6}}\,\|f\|_{L^2(M)}\|g\|_{L^2(M)}\|h\|_{L^2(M)} \tag{1.6}$$

For $n=2$, the bilinear and trilinear estimates imply the estimate (1.4), respectively for $q=4$ and $q=6$. Moreover, (1.6) implies (1.5) for Neumann conditions by taking h constant. For $n=3$ this is no longer the case. However, the bilinear estimate for $n=3$ implies (up to a logarithmic loss) the best possible L^4 linear estimate for manifolds with Lipschitz metric, by the examples of [8], and our proof in fact establishes Theorem 1.1 in this context.

As in [9], a key step is to work on the double $\tilde M$ of the manifold M, obtained by attaching two copies of M along the boundary, and taking coordinate patches along $\partial M\subset\tilde M$ which agree with geodesic normal coordinates (x',x_n) on each copy of M. In these coordinates, the lift $\tilde{\mathrm{g}}$ of g to $\tilde M$ is given by $\mathrm{g}^{ij}(x',|x_n|)$, and hence $\tilde{\mathrm{g}}$ extends to $\tilde M$ with a Lipschitz type singularity along ∂M. Dirichlet and Neumann eigenfunctions on M correspond to eigenfunctions on $\tilde M$ which are, respectively, odd or even under $x_n\to -x_n$. Hence, L^q bounds on spectral clusters, and the multilinear analogs we consider, can be obtained by proving the same bounds on $(\tilde M,\tilde{\mathrm{g}})$.

The linear estimates of [9] were obtained by establishing mixed-norm $L^q_xL^2_t$ estimates on $\tilde M$ for the evolution of a spectral cluster under the wave equation, and we follow a similar approach here. In that paper, the precise nature of the singularity of $\tilde{\mathrm{g}}$ along ∂M was used, and a microlocal decomposition of the cluster was made in terms of angle from tangent to ∂M. Estimates were obtained over small slabs, with size depending on the frequency and angle, and summing over slabs led to a frequency dependent loss for the estimates on $\tilde M$.

In contrast, the results of this paper go through generally for the case of a boundary-free Riemmanian manifold with metric of Lipschitz regularity, as with the linear spectral cluster estimates of [7], or the Strichartz estimates of Tataru

[11]. As in those papers, we obtain estimates over small slabs with size depending on the frequency, and use a rescaling argument to reduce matters to obtaining estimates for C^2 metrics. We then use wave packet methods to obtain dispersive estimates, as in [5], [6] and [11], Summing over slabs then leads to a frequency dependent loss.

2. Microlocal Reductions

For the remainder of this paper, we assume M is a compact manifold without boundary, and g is a metric of Lipschitz regularity. The condition that f, g, h be spectrally localized can be relaxed, and we work instead with a *quasimode* condition. We state the condition for f here, the condition for g and h being analogous. For each local coordinate chart we write

$$\mathrm{g}\, d^2 f + \lambda^2 f = w\,, \qquad \mathrm{g}\, d^2 = \sum_{j,k} \mathrm{g}^{jk}(x)\, \partial_j \partial_k\,. \tag{2.1}$$

Given such an equation, we set

$$|||f|||_\lambda = \|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \lambda^{-2}\|d^2 f\|_{L^2} + \lambda^{-1}\|w\|_{L^2}\,.$$

If $f = \chi_\lambda f$, then for ϕ a smooth cutoff to a local coordinate system, the function ϕf satisfies the equation (2.1) on $\mathbb{R}^n$, and

$$|||\phi f|||_\lambda \lesssim \|f\|_{L^2(M)}\,.$$

For the bilinear estimates it thus suffices to prove, for each coordinate chart, that

$$\|\phi f\, \phi g\|_{L^2} \lesssim \Lambda(\mu)\, |||\phi f|||_\lambda\, |||\phi g|||_\mu\,,$$

and analogously for the trilinear version.

By choosing appropriate coordinates, we may assume f satisfies (2.1) on $\mathbb{R}^n$, with f supported in the unit ball, and

$$\|\mathrm{g}^{ij} - \delta^{ij}\|_{\mathrm{Lip}(\mathbb{R}^n)} \le c_0\,,$$

with c_0 a constant to be chosen suitably small.

Let $S_r = S_r(D)$ denote a smooth cutoff on the Fourier transform side to frequencies of size $|\xi| \le r$. Let $\mathrm{g}_\lambda = S_{c^2\lambda}\mathrm{g}$, for c to be chosen suitably small. Then

$$\|(\mathrm{g} - \mathrm{g}_\lambda) d^2 f\|_{L^2} \lesssim c^{-2}\lambda^{-1}\|d^2 f\|_{L^2}\,,$$

and thus we may replace g by g_λ in (2.1) at the expense of absorbing the above term into w, which does not change the size of $|||f|||_\lambda$.

We next take a microlocal partition of unity, $1 = \sum \Gamma(D)$, where each $\Gamma(\xi)$ is a smooth symbol of order 0 supported in a cone of small angle. By the Coifman-Meyer commutator theorem [3], since g is Lipschitz

$$[\mathrm{g}, \Gamma(D)]\, d\, :\, L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,,$$

hence $\Gamma(D) f$ satisfies the equation (2.1), with $|||\Gamma(D) f|||_\lambda \lesssim |||f|||_\lambda\,.$

Since there are finitely many terms, we may replace f by $\Gamma(D) f$, which is no longer compactly supported, but is rapidly decreasing and smooth outside the unit ball. Without loss of generality we assume that $\Gamma(\xi)$ is supported within a small angle of the ξ_1 axis. We similarly replace g and h by $\Gamma'(D) g$ and $\Gamma''(D) h$, localized in frequency to small cones along general directions.

Letting $x' = x_2$ in case of dimension $n = 2$, and $x' = (x_2, x_3)$ in case of dimension $n = 3$, we bound

$$\|fg\|_{L^2} \leq \|f\|_{L^\infty_{x_1} L^2_{x'}} \|g\|_{L^2_{x_1} L^\infty_{x'}}$$

$$\|fgh\|_{L^2} \leq \|f\|_{L^\infty_{x_1} L^2_{x'}} \|g\|_{L^4_{x_1} L^\infty_{x'}} \|h\|_{L^4_{x_1} L^\infty_{x'}}$$

Since $\Gamma(D)f$ is rapidly decreasing outside the unit ball, it suffices to take the norms above over the ball of radius 2. Theorem 1.1 is then a result of the following.

Theorem 2.1. *Suppose that f satisfies the equation*

$$\mathrm{g}_\lambda d^2 f + \lambda^2 f = w \tag{2.2}$$

Then the following hold, where the norms on the left side are over a bounded set

$$\|f\|_{L^2_{x_1} L^\infty_{x'}} \lesssim \lambda^{\frac{2}{3}} (\log \lambda)^{\frac{1}{2}} \, |||f|||_\lambda \,, \qquad n = 3 \tag{2.3}$$

$$\|f\|_{L^4_{x_1} L^\infty_{x'}} \lesssim \begin{cases} \lambda^{\frac{1}{3}} \, |||f|||_\lambda \,, & n = 2 \\ \lambda^{\frac{5}{6}} \, |||f|||_\lambda \,, & n = 3 \end{cases} \tag{2.4}$$

Futhermore, if $\hat{f}(\xi)$ is supported in a small cone about the ξ_1 axis, then

$$\|f\|_{L^\infty_{x_1} L^2_{x'}} \lesssim |||f|||_\lambda \tag{2.5}$$

Proof. We start by localizing f dyadically in frequency. Let

$$f = S_{c\lambda} f + (S_{c^{-1}\lambda} - S_{c\lambda}) f + (1 - S_{c^{-1}\lambda}) f \equiv f_{<\lambda} + f_\lambda + f_{>\lambda} \,.$$

Since $[S_{c\lambda}, \mathrm{g}_\lambda]\, d : L^2 \to L^2$, f_λ satisfies (2.2), with $|||w_\lambda|||_\lambda \lesssim |||f|||_\lambda$, and similarly for $f_{>\lambda}$ and $f_{<\lambda}$. Furthermore, by the frequency localization of g_λ, each of w_λ, $w_{<\lambda}$, and $w_{>\lambda}$ is also localized to the appropriate range of frequencies.

A simple integration by parts argument (see the proof of Corollary 5 of [7]) yields that, for c sufficiently small,

$$\lambda \|f_{<\lambda}\|_{L^2(\mathbb{R}^n)} + \|df_{>\lambda}\|_{L^2(\mathbb{R}^n)} \lesssim |||f|||_\lambda \,.$$

Elliptic regularity additionally gives the bound

$$\|d^2 f_{>\lambda}\|_{L^2(\mathbb{R}^n)} \lesssim \lambda \, |||f|||_\lambda \,.$$

Sobolev embedding then yields each of the estimates (2.3)–(2.5) for $f_{<\lambda}$ and $f_{>\lambda}$. Indeed, there is a gain of $\lambda^{\frac{2}{3}} (\log \lambda)^{\frac{1}{2}}$ in the estimate (2.3), and a gain of $\lambda^{\frac{7}{12}}$ in the estimate (2.4), for these terms.

Consequently, we are reduced to establishing (2.3)–(2.5) for the term f_λ. We start with (2.5). Let V denote the vector field

$$V = 2(\partial_1 f_\lambda)\, \mathrm{g}_\lambda \, df_\lambda + \big(\lambda^2 f_\lambda^2 - \langle \mathrm{g}_\lambda \, df_\lambda, df_\lambda \rangle\big) \overrightarrow{e_1} \,.$$

Then

$$\operatorname{div} V = 2(\partial_1 f_\lambda)\, (\operatorname{div} \mathrm{g}_\lambda) \cdot df_\lambda + 2(\partial_1 f_\lambda)\, w_\lambda - \langle (\partial_1 \mathrm{g}_\lambda) df_\lambda, df_\lambda \rangle \,.$$

Applying the divergence theorem on the set $x_1 \leq r$ yields

$$\int_{x_1 = r} V_1 \, dx' \lesssim \lambda^2 \|f_\lambda\|^2_{L^2(\mathbb{R}^n)} + \|df_\lambda\|^2_{L^2(\mathbb{R}^n)} + \|w_\lambda\|^2_{L^2(\mathbb{R}^n)} \,.$$

Since g_λ is pointwise close to the flat metric, we have pointwise that

$$V_1 \geq \tfrac{3}{4}|\partial_1 f_\lambda|^2 + \tfrac{3}{4}\lambda^2|f_\lambda|^2 - |\partial_{x'} f_\lambda|^2 .$$

The frequency localization of $\widehat{f_\lambda}$ to $|\xi'| \leq c\lambda$ yields

$$\int_{x_1=r} V_1\, dx' \geq \frac{1}{2}\int_{x_1=r} |df_\lambda|^2 + \lambda^2 |f_\lambda|^2\, dx' .$$

Consequently,

$$\begin{aligned}\lambda^{-1}\|df_\lambda\|_{L^\infty_{x_1}L^2_{x'}} + \|f_\lambda\|_{L^\infty_{x_1}L^2_{x'}} &\lesssim \|f_\lambda\|_{L^2(\mathbb{R}^n)} + \lambda^{-1}\|df_\lambda\|_{L^2(\mathbb{R}^n)} + \lambda^{-1}\|w_\lambda\|_{L^2(\mathbb{R}^n)} \\ &\leq |||f_\lambda|||_\lambda ,\end{aligned}$$

yielding estimate (2.5).

For the remainder, we assume that $\hat{f}(\xi)$ is localized to a small cone along the direction ω. In this case, the above argument yields uniform L^2 bounds over hyperplanes of the form $\omega \cdot x = r$. In the proof of (2.3)–(2.4) we will use the following consequence. Suppose that S_R is a slab of the form $\omega \cdot x \in I$, where I is an interval of length $|I| = R$. Then

$$\lambda^{-1}\|df_\lambda\|_{L^2(S_R)} + \|f_\lambda\|_{L^2(S_R)} \lesssim R^{\frac{1}{2}} |||f_\lambda|||_\lambda . \tag{2.6}$$

We cover the bounded set on which the norms in (2.3) and (2.4) are taken by $\approx \lambda^{\frac{1}{3}}$ slabs of the form $\omega \cdot x \in I$, where $|I| = R = \lambda^{-\frac{1}{3}}$. Then

$$\|f_\lambda\|_{L^p_{x_1}L^\infty_{x'}} \lesssim \lambda^{\frac{1}{3p}} \sup_{S_R} \|f_\lambda\|_{L^p_{x_1}L^\infty_{x'}(S_R)} . \tag{2.7}$$

We will establish the following result. Suppose that Q_R is a cube of sidelength $R = \lambda^{-\frac{1}{3}}$, and Q_R^* its double. Then

$$\begin{aligned}&\|f_\lambda\|_{L^p_{x_1}L^\infty_{x'}(Q_R)} \\ &\quad\lesssim c_p(\lambda) R^{-\frac{1}{2}}\big(\|f_\lambda\|_{L^2(Q_R^*)} + \lambda^{-1}\|df_\lambda\|_{L^2(Q_R^*)} + R\lambda^{-1}\|w_\lambda\|_{L^2(Q_R^*)}\big)\end{aligned} \tag{2.8}$$

where

$$c_2(\lambda) = \lambda^{\frac{1}{2}}(\log\lambda)^{\frac{1}{2}} , \quad n = 3 \qquad\qquad c_4(\lambda) = \begin{cases} \lambda^{\frac{1}{4}} , & n = 2 \\ \lambda^{\frac{3}{4}} , & n = 3 \end{cases}$$

If we cover the slab S_R by disjoint cubes Q_R, then by (2.6) we obtain

$$\|f_\lambda\|_{L^p_{x_1}L^\infty_{x'}(S_R)} \lesssim c_p(\lambda)\, |||f_\lambda|||_\lambda ,$$

and (2.7) yields (2.3)–(2.4).

The estimate (2.8) is scale-invariant. Precisely, if we change $x \to Rx$, so that Q becomes a cube of size 1, and $f_\lambda(R\,\cdot)$ is frequency localized at scale $R\lambda = \lambda^{\frac{2}{3}}$, then, with $\mu = \lambda^{\frac{2}{3}}$, estimate (2.8) is equivalent to the following

$$\|f_\mu\|_{L^p_{x_1}L^\infty_{x'}(Q)} \lesssim c_p(\mu)\big(\|f_\mu\|_{L^2(Q^*)} + \mu^{-1}\|df_\mu\|_{L^2(Q^*)} + \mu^{-1}\|w_\mu\|_{L^2(Q^*)}\big) . \tag{2.9}$$

Here, $f_\mu(x) = f_\lambda(\lambda^{-\frac{1}{3}}x)$, which satisfies the equation

$$g_\mu d^2 f_\mu + \mu^2 f_\mu = w_\mu ,$$

with $g_\mu(x) = g_\lambda(\lambda^{-\frac{1}{3}}x)$. Observe that

$$\|dg_\mu\|_{L^\infty} \leq c_0\lambda^{-\frac{1}{3}} = c_0\mu^{-\frac{1}{2}} , \tag{2.10}$$

hence

$$\|\mathrm{g}_\mu - \mathrm{g}_{\mu^{1/2}}\|_{L^\infty} \le c_0\mu^{-1}\,,$$

with $\mathrm{g}_{\mu^{1/2}} = S_{c^2\mu^{1/2}}\mathrm{g}_\mu$. Thus f_μ satisfies the equation

$$\mathrm{g}_{\mu^{1/2}} d^2 f_\mu + \mu^2 f_\mu = w_\mu\,, \tag{2.11}$$

with the right-hand side of (2.9) of comparable size.

It follows from (2.10) that the metric $\mathrm{g}_{\mu^{1/2}}$ is of regularity C^2. Indeed,

$$\|\mathrm{g}^{jk}_{\mu^{1/2}} - \delta^{jk}\|_{C^2} \le c_0\,.$$

To establish (2.9), we may thus use techniques developed to establish dispersive estimates for operators of principal type with C^2 coefficients. We follow below the path through squarefunction estimates for solutions to a first order hyperbolic equation, by introducing a time variable, as in [6]. It should also be possible to establish the dispersive estimates directly for (2.11) using methods of [5].

Let

$$p(\,\cdot\,,\xi) = S_{c^2\mu^{1/2}}\Bigl(\sum_{j,k}\mathrm{g}^{jk}_{\mu^{1/2}}(\cdot)\,\xi_j\,\xi_k\Bigr)^{\frac12}\,.$$

Then

$$\|p(x,D)^2 f_\mu + \mathrm{g}_{\mu^{1/2}} d^2 f_\mu\|_{L^2(\mathbb{R}^n)} \lesssim \mu\,\|f_\mu\|_{L^2}\,.$$

Thus,

$$\bigl(\mu + p(x,D)\bigr)\bigl(\mu - p(x,D)\bigr) f_\mu = w_\mu\,,$$

with the error harmlessly absorbed into w_μ. The operator $\mu + p(x,D)$ is elliptic on the frequency support of $\bigl(\mu - p(x,D)\bigr)f_\mu$, hence we may write

$$\bigl(\mu - p(x,D)\bigr) f_\mu = \mu^{-1} w_\mu\,,$$

with $|||f|||_\lambda$ still of comparable size. Finally, let

$$u(t,x) = e^{-it\mu} f_\mu(x)\,, \qquad\qquad F = \mu^{-1} e^{-it\mu} w_\mu\,.$$

Then

$$\bigl(\partial_t + ip(x,D)\bigr)u = F\,,$$

and it suffices to show that

$$\|u\|_{L^p_{x_1}L^\infty_{x'}L^2_t(Q\times[0,1])} \lesssim c_p(\mu)\bigl(\,\|u\|_{L^\infty_t L^2_x([0,1]\times\mathbb{R}^n)} + \|F\|_{L^1_t L^2_x([0,1]\times\mathbb{R}^n)}\bigr)\,. \tag{2.12}$$

Our proof of (2.12) follows very closely the proof of the linear spectral cluster estimates in [6]; we outline just the main steps here. Following [6, §3], consider the wave packet transform of u_μ,

$$\bigl(T_\mu u\bigr)(t,x,\xi) = \mu^{\frac n4}\int e^{-i\langle\xi, z-x\rangle}\,\phi\bigl(\mu^{\frac12}(z-x)\bigr)\,u(t,z)\,dz\,,$$

where ϕ is a real, even Schwartz function, with $\|\phi\|_{L^2} = (2\pi)^{-\frac n2}$, and with Fourier transform supported in the unit ball $\{|\xi|\le 1\}$. Then

$$\partial_t T_\mu u(t,x,\xi) = \Bigl(d_\xi p(x,\xi)\cdot d_x - d_x p(x,\xi)\cdot d_\xi\Bigr) T_\mu u(t,x,\xi) + G(t,x,\xi)\,,$$

where $G(t,x,\xi) = 0$ unless $\frac18\mu < |\xi| < 2\mu$, and

$$\|G\|_{L^1_t L^2_{x,\xi}} \lesssim \|u\|_{L^1_t L^2_x} + \|F\|_{L^1_t L^2_x}\,.$$

Let χ_t denote the canonical transform on $\mathbb{R}^{2n}_{x,\xi} = T^*(\mathbb{R}^n)$ generated by the Hamiltonian flow of p. Thus, $\chi_t(x,\xi) = \gamma(t)$, where γ is the integral curve with $\gamma(0) = (x,\xi)$. Then we have

$$\big(T_\mu u\big)(t,x,\xi) = \big(T_\mu u\big)(0,\chi_{-t}(x,\xi)) + \int_0^t G(r,\chi_{r-t}(x,\xi))\,dr\,.$$

Thus, $T_\mu u(t,x,\xi)$ is an integrable superposition over r of $1_{t>r}$ multiplied by a function invariant under the Hamiltonian flow of p.

Since $u(t,x) = T_\mu^*\big(T_\mu u\big)(t,x,\xi)$, it suffices to show

$$\|W\tilde f\|_{L^p_{x_1}L^\infty_{x'}L^2_t(\mathbb{R}^n\times[0,1])} \lesssim c_p(\mu)\|f\|_{L^2_{x,\xi}}\,, \tag{2.13}$$

where

$$\big(W\tilde f\,\big)(t,x) = T_\mu^*\big(\tilde f\circ\chi_{-t}\big)(x)\,.$$

This is in turn equivalent to the following bounds

$$\|WW^*F\|_{L^p_{x_1}L^\infty_{x'}L^2_t(Q\times[0,1])} \lesssim c_p(\mu)^2\,\|F\|_{L^{p'}_{x_1}L^1_{x'}L^2_t(Q\times[0,1])}\,. \tag{2.14}$$

The operator WW^* has an integral kernel K which is highly localized to a μ^{-1} neighborhood of the light cone, with the dispersive rate of decay away from the origin, see [6, (3.11)],

$$|K(s,y;t,z)| \lesssim \mu^n\big(1+\mu\,|y_1-z_1|\,\big)^{-\frac{n-1}{2}}\,\big(1+\mu\,\big|\,d(y,z)-|s-t|\big|\,\big)^{-N}\,, \tag{2.15}$$

with $d(y,z)$ the distance of y to z determined by p.

We remark that in [6] this estimate was established assuming the kernel was microlocalized near the ξ_1 axis. That assumption, however, was necessary for L^2-energy estimates, not the above dispersive estimates. Indeed, the proof of [6, (3.11)] establishes (2.15) with $|y_1-z_1|$ replaced by $|y-z|$, since $|t-s|\approx|y-z|$ on the light cone, and hence holds without any assumption of conic micro-localization.

Estimate (2.15) implies that, for each (y_1,z_1),

$$\Big\|\int K(s,y;t,z)\,v(t,z')\,dt\,dz'\Big\|_{L^\infty_{y'}L^2_s} \lesssim \mu^{n-1}(1+\mu\,|y_1-z_1|)^{-\frac{n-1}{2}}\,\|v\|_{L^1_{z'}L^2_t}\,.$$

For $n=3$ and $p=2$, estimate (2.14) follows from

$$\int_{|z_1|\le 2}\mu^2(\,1+\mu\,|y_1-z_1|\,)^{-1} \le \mu\log\mu\,.$$

For $p=4$, (2.14) follows from the Hardy-Litlewood-Sobolev inequality, together with the bound

$$\mu^{n-1}(\,1+\mu\,|y_1-z_1|\,)^{-\frac{n-1}{2}} \le \mu^{n-\frac32}|y_1-z_1|^{-\frac12}\,. \qquad \square$$

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