Strichartz estimates in polygonal domains and cones

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Joint work with:

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The wave equation on $\mathbb{R}^n$

- Initial value problem for the wave equation

$$\Box u := (D_t^2 - \Delta)u = 0, \qquad (u, \partial_t u)\big|_{t=0} = (f, g),$$

$$u(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \qquad (D_t = -i\partial_t, \quad \Delta \geq 0)$$

- Properties:

$$\|\nabla_{t,x} u(t,\cdot)\|_{L^2}^2 = \|\nabla_{t,x} u(0,\cdot)\|_{L^2}^2 \qquad \text{(energy conservation)}$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C(u)(1+|t|)^{-\frac{n-1}{2}} \qquad \text{(decay inequality)}$$

Nonlinear wave equations

- Semilinear wave equation with power type nonlinearity

$$\Box u = \pm |u|^{r-1} u$$

- Inhomogeneous energy estimates

$$\|\nabla_{t,x} u(t,\cdot)\|_{L^2} \lesssim \|\nabla_{t,x} u(0,\cdot)\|_{L^2} + \int_0^t \|\Box u(s,\cdot)\|_{L^2}\, ds$$

- In order to linearize the equation, need to estimate *powers* of solutions efficiently

$$\|u^r\|_{L^1(I;L^2(\mathbb{R}^n))} = \|u\|^r_{L^r(I;L^{2r}(\mathbb{R}^n))}, \qquad I = (-T, T)$$

Strichartz estimates

- Robert Strichartz (1970's)–estimates for $\Box u = 0$:

$$\|u\|_{L^q(\mathbb{R}^{n+1})} \leq C\left(\|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^n)}\right), \qquad q = \frac{2(n+1)}{n-1}$$

- Consequence of Stein-Tomas restriction theorem: $\widehat{u}(\tau,\xi)$ is supported on the cone $S = \{\tau^2 = |\xi|^2\}$,

$$\|u\|_{L^q(\mathbb{R}^{n+1})} \leq C\|\widehat{u}\|_{L^2(S)}$$

which is dual to a Fourier restriction estimate

Strichartz estimates

- 80's/90's: Ginibre-Velo, Lindblad-Sogge, Keel-Tao, others

$$\|u\|_{L^p(\mathbb{R};L^q(\mathbb{R}^n))} \leq C\left(\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}\right)$$

- Admissibility conditions:

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \qquad \text{(Scaling)}$$

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \qquad \text{(Knapp example/Lorentz)}$$

Littlewood-Paley decompostions

- Take a Littlewood-Paley decomposition in the spatial frequencies

$$u = \sum_{k=-\infty}^{\infty} u_k, \quad u_k(t,\cdot) = \mathcal{F}^{-1}\{\beta_k(\xi)\widehat{u}(t,\xi)\},$$

$$\operatorname{supp}(\beta_k) \subset \left\{2^{k-\frac{1}{2}} < |\xi| < 2^{k+\frac{3}{2}}\right\}, \qquad \sum_{k=-\infty}^{\infty} \beta_k(\xi) = 1$$

- The Littlewood-Paley squarefunction estimate reduces matters to

$$\|u_k\|_{L^p(L^q)} \lesssim 2^{\gamma k}\|f_k\|_{L^2} + 2^{\gamma(k-1)}\|g_k\|_{L^2} \qquad k \in \mathbb{Z}$$

- Use scale invariance $(t,x) \mapsto (2^{-k}t, 2^{-k}x)$ to reduce to

$$\|u_0\|_{L^p(L^q)} \lesssim \|f_0\|_{L^2} + \|g_0\|_{L^2} \qquad (k=0)$$

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Frequency localized estimates

- Crucial matter: show that

$$\|u_0(t,\cdot)\|_{L^\infty} \lesssim (1+|t|)^{-\frac{n-1}{2}} \left(\|f_0\|_{L^1} + \|g_0\|_{L^1}\right)$$

- Oscillatory integral approach is most effective

$$\left| \int e^{i(x-y)\cdot\xi \pm it|\xi|} \alpha(|\xi|)\, d\xi \right| \lesssim (1+|t|)^{-\frac{n-1}{2}}, \qquad \alpha \in C_c^\infty(\mathbb{R}_+)$$

- Can view the Littlewood-Paley multiplier as an operator which regularizes the Schwartz (distributional) kernels of

$$\frac{\sin(t\sqrt{\Delta})}{\sqrt{\Delta}} \quad \text{and} \quad \cos(t\sqrt{\Delta})$$

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Boundary value problems

- Let Ω be a domain in $\mathbb{R}^n$, and consider solutions to

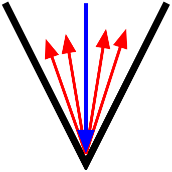
$$(D_t^2 - \Delta)u = 0, \qquad (u, \partial_t u)\big|_{t=0} = (f, g),$$

$$u(t,\cdot)|_{\partial\Omega} = 0 \text{ (Dirichlet)} \qquad \text{or} \qquad \frac{\partial u}{\partial \nu}(t,\cdot)\Big|_{\partial\Omega} = 0 \text{ (Neumann)}$$

- Boundary conditions affect the flow of energy
- Trapped rays can preclude a global (in time) estimate

Boundary value problems

- Partial progress on smooth boundaries: Smith-Sogge, Burq-Lebeau-Planchon, MDB-Smith-Sogge
- Common thread–can construct a parametrix for the equation
- Domains with corners? No known effective parametrix
  - Melrose-Vasy-Wunsch: If a singularity lies on a ray which approaches a corner, it lies within the union of a family of rays after the interaction



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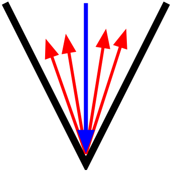

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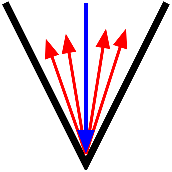
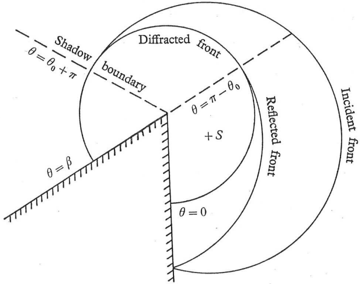


Sommerfeld's example

Sommerfeld (1896) did explicit computations in the exterior of a wedge–he showed that when a wavefront interacts with the tip, a spherical wave of singularities is formed, even into the shadow region



(Figure from Friedlander's *Sound Pulses*)

Main theorem for domains

Theorem (MDB, Ford, Marzuola)

Let Ω be a domain in $\mathbb{R}^2$ whose boundary consists of a finite number of line segments. Then any solution to the wave equation with Dirichlet or Neumann BC's satisfies

$$\|u\|_{L^p((-T,T);L^q(\Omega))} \lesssim \|f\|_{H^\gamma(\Omega)} + \|g\|_{H^{\gamma-1}(\Omega)}$$

$$\frac{1}{p} + \frac{2}{q} = 1 - \gamma \qquad \text{(scaling)}$$

$$\frac{2}{p} + \frac{1}{q} \leq \frac{1}{2} \qquad \text{(Knapp admissability)}$$

Doubling the domain

- Since the estimate is local in time, finite speed of propagation means that it suffices to work locally in space, that is, over sets as small as you like
- Away from the vertices: use the method of images
- Near the vertices: impose polar coordinates (r, θ) centered at the vertex. If the angle is α, $(0, \delta) \times [0, \alpha] \subset \mathbb{R}_+ \times \mathbb{S}^1$ will describe the neighborhood.
- This neighborhood can be “doubled" by gluing a copy of the corner on to the original

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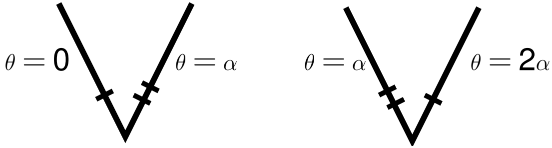

- Doubling gives $(0,\delta) \times \mathbb{R}/2\alpha$ equipped with the metric $dr^2 + r^2 d\theta^2$, a subset of the Euclidean cone
- $C(\mathbb{S}^1_\rho) = \mathbb{R}_+ \times \mathbb{R}/2\pi\rho$, the Euclidean cone of radius ρ $(\rho = \alpha/\pi)$. It has the flat metric $\mathrm{g} = dr^2 + r^2 d\theta^2$

Doubling the domain

- Dirichlet solutions can be extended by writing

$$u(t,r,\theta) = \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{\infty} u_j(t,r) \sin\left(\frac{j\pi\theta}{\alpha}\right)$$

- Neumann solutions can be extended by writing

$$u(t,r,\theta) = u_0(t,r) + \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{\infty} u_j(t,r) \cos\left(\frac{j\pi\theta}{\alpha}\right)$$

Main theorem for cones

Theorem (MDB, Ford, Marzuola)

Let $C(\mathbb{S}^1_\rho)$ be the Euclidean cone of radius $\rho > 0$. Then for any admissible triple (p, q, γ)

$$\|u\|_{L^p(\mathbb{R};L^q(C(\mathbb{S}^1_\rho)))} \lesssim \|f\|_{\dot{H}^\gamma(C(\mathbb{S}^1_\rho))} + \|g\|_{\dot{H}^{\gamma-1}(C(\mathbb{S}^1_\rho))}$$

- On $C(\mathbb{S}^1_\rho)$, wave equation involves the Laplace-Beltrami operator

$$-\Delta_{\mathrm{g}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

The Spectral Theorem

Spectral Theorem

There exists a measure space (Y, μ) and a unitary map $W : L^2(Y, \mu) \to L^2(C(\mathbb{S}^1_\rho))$ and a measurable function $a(y)$ on Y such that

$$W^{-1}\Delta_g W\, h(y) = a(y)h(y), \qquad \text{whenever } W\, h \in \text{Dom}(\Delta_g).$$

Furthermore, functions $f(\Delta_g)$ can be defined by

$$W^{-1} f(\Delta_g) W\, g(y) = f(a(y))g(y)$$

Can take a Littlewood-Paley decomp. w.r.t. the spectrum of Δ_g

$$I = \sum_{k=-\infty}^{\infty} \beta_k(\sqrt{\Delta_g})$$

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Functional calculus on cones

- Begin with separated solutions to the Helmholtz eqn $(\Delta_g - \lambda^2)g(r)\varphi_\nu(\theta) = 0$, with $-\varphi_\nu''(\theta) = \nu^2\varphi_\nu(\theta)$
- $g(r)$ must satisfy the Bessel-type equation

$$L_\nu g = -g''(r) - \frac{1}{r}g'(r) + \frac{\nu^2}{r^2}g(r) = \lambda^2 g(r) \;\Rightarrow\; g(r) = \mathcal{C}_\nu(\lambda r)$$

- Taking $\mathcal{C}_\nu(\lambda r) = J_\nu(\lambda r)$, define the Hankel transform

$$H_\nu(g)(\lambda) = \int_0^\infty g(r) J_\nu(\lambda r) r\, dr$$

- H_ν defines a unitary map $H_\nu : L^2(\mathbb{R}_+, r\,dr) \to L^2(\mathbb{R}_+, \lambda\, d\lambda)$ and $H_\nu \circ H_\nu = I$, $H_\nu(L_\nu g)(\lambda) = \lambda^2 H_\nu(g)(\lambda)$

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Functional calculus on cones

- Use this to create a spectral representation of Δ_g, Schwartz kernel of $f(\Delta_g)$ will have the form

$$K_f(r_1,\theta_1;r_2,\theta_2)=\sum_\nu \widetilde{K}_f(r_1,r_2,\nu)\varphi_\nu(\theta_1)\overline{\varphi_\nu(\theta_2)}$$

where ν indexes an O.N. basis of eigenfunctions and

$$\widetilde{K}_f(r_1,r_2,\nu)=\int_0^\infty f(\lambda^2)J_\nu(\lambda r_1)J_\nu(\lambda r_2)\lambda\,d\lambda$$

- Use this to understand kernels of $e^{-it\sqrt{\Delta_g}}$

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Formulae for the coefficients

- Lipschitz-Hankel integral
  ($Q_{\nu-\frac{1}{2}}$ = Legendre function of the 2nd kind, order $\nu - \frac{1}{2}$)

$$\int_0^\infty e^{-it\lambda} J_\nu(\lambda r_1) J_\nu(\lambda r_2)\, d\lambda = \frac{1}{\pi}(r_1 r_2)^{-\frac{1}{2}} Q_{\nu-\frac{1}{2}}\left(\frac{r_1^2 + r_2^2 - t^2}{2 r_1 r_2}\right)$$

- Now sum to obtain formulae for $\sin(t\sqrt{\Delta_g})/\sqrt{\Delta_g}$, $\cos(t\sqrt{\Delta_g})$

$$K_f(r_1, \theta_1; r_2, \theta_2) = \frac{1}{2\pi\rho} \sum_{j=-\infty}^{\infty} \widetilde{K}_f\left(r_1, r_2, \frac{|j|}{\rho}\right) \exp\left(\frac{ij(\theta_1 - \theta_2)}{\rho}\right)$$

Formulae for the solution operators

- Cheeger-Taylor: formulas for the kernel of $\sin(t\sqrt{\Delta_g})/\sqrt{\Delta_g}$ on cones
- MDB-Ford-Marzuola: formulas for $\cos(t\sqrt{\Delta_g})$ when $\rho < 1$
- Kernels above take the form

$$K_{geom}(r_1, \theta_1; r_2, \theta_2) + K_{diff}(r_1, \theta_1; r_2, \theta_2)$$

  - "K_{geom}" consists of terms arising from a formal application of the method of images
  - "K_{diff}" arises from diffraction by the cone tip

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Good news and bad news

- Littlewood-Paley works as before and the wave equation is invariant under dilations
- Problem: Have good formulae for e.g. $\sin(t\sqrt{\Delta_g})/\sqrt{\Delta_g}$, but not

$$\frac{\sin(t\sqrt{\Delta_g})}{\sqrt{\Delta_g}}\beta_0(\sqrt{\Delta_g})$$

- Very difficult to obtain oscillatory integrals analogous to those on $\mathbb{R}^2$
- Take a new perspective on the problem in $\mathbb{R}^2$ and regularize the kernel of $\sin(t\sqrt{\Delta})/\sqrt{\Delta}$,

$$K(t,x,y) = \pi^{-1}(t^2 - |x-y|^2)_+^{-\frac{1}{2}}$$

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The averaging approach on $\mathbb{R}^2$

- Treat Littlewood Paley operator as a regularizing operator

$$\int K(t,x,y)\beta_0(\sqrt{\Delta})g_0(y)\,dy = \int \left(\beta_0(\sqrt{\Delta_y})K(t,x,y)\right) g_0(y)\,dy$$

- On $\mathbb{R}^2$, convolution kernel of $\beta_0(\sqrt{\Delta})$ is a Schwartz function, rapidly decreasing on the unit scale
- Morally, $|\beta_0(\sqrt{\Delta_y})K(t,x,y)|$ is controlled by its average on a set of size one
- Averages are bounded since $(t^2-r^2)^{-\frac{1}{2}} \leq t^{-\frac{1}{2}}(t-r)^{-\frac{1}{2}}$ and the second factor is integrable in r

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- Morally, $|\beta_0(\sqrt{\Delta_y})K(t,x,y)|$ is controlled by its average on a set of size one
- Averages are bounded since $(t^2 - r^2)^{-\frac{1}{2}} \leq t^{-\frac{1}{2}}(t-r)^{-\frac{1}{2}}$ and the second factor is integrable in r

The averaging approach on $\mathbb{R}^2$

- Treat Littlewood Paley operator as a regularizing operator

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The averaging approach on the Euclidean cone

- Heat kernel results give bounds on the kernel of $\beta_0(\sqrt{\Delta_g})$ in terms of the distance function on $C(\mathbb{S}^1_\rho)$), yields similar control via averages
- Behavior of the geometric term is similar to the corresponding propagator on $\mathbb{R}^2 \Rightarrow$ averaging approach carries over to the cone
- We prove pointwise bounds on the diffractive term that display a similar character

$$|K_{diff}(r_1, \theta_1; r_2, \theta_2)| \leq (t^2 - (r_1 + r_2)^2)_+^{-\frac{1}{2}}$$

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