Integration of differential forms and reparameterization of surfaces

Theorem 0.1. *Suppose $\phi : V \to M$, $\psi : U \to M$ are parameterizations of an n-surface $M \subset \mathbb{R}^m$ with ϕ, ψ both bijections so that $F = \phi^{-1} \circ \psi : U \to V$ is well defined bijection. Assume further that U, V are connected. If U, F satisfy the hypotheses of the hypotheses of the change of variables theorem (e.g. either Theorem 12.46 or 12.65 in Wade), then given a differential form ω defined on M*

$$\int_V \omega_{\phi(y)} \left(\frac{\partial \phi}{\partial y_1}, \ldots, \frac{\partial \phi}{\partial y_n} \right) dy = \pm \int_U \omega_{\psi(z)} \left(\frac{\partial \psi}{\partial z_1}, \ldots, \frac{\partial \psi}{\partial z_n} \right) dz \tag{0.1}$$

where $\pm$ is the sign of the determinant of $F'(z)$ (which cannot change on U since it is connected). Hence given an orientation on M, $\int_M \omega$ is defined independently of the choice of parameterization.

Proof. By theorem (the "fact" in class) $\omega_p = \sum_I c_I(p) dx_I$. By linearity, it suffices to restrict attention to one term in the sum. Moreover, we may assume that this term corresponds to the index $I = (1, \ldots, n)$, as it is a change of notation otherwise. We thus assume without loss of generality that $\omega_p = f(p)\, dx_1 \wedge \cdots \wedge dx_n$ for some smooth function $f(p)$.

Define $\pi : \mathbb{R}^m \to \mathbb{R}^n$ by $\pi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n)$, that is, π is projection onto the first n coordinates. Hence $\pi \circ \phi(y) = (\phi_1(y), \ldots, \phi_n(y))$. Now observe that

$$\begin{aligned} dx_1 \wedge \cdots \wedge dx_n \left(\frac{\partial \phi}{\partial y_1}, \ldots, \frac{\partial \phi}{\partial y_n} \right) &= \det \begin{bmatrix} \frac{\partial \phi_1}{\partial y_1} & \frac{\partial \phi_2}{\partial y_1} & \cdots & \frac{\partial \phi_n}{\partial y_1} \\ \frac{\partial \phi_1}{\partial y_2} & \frac{\partial \phi_2}{\partial y_2} & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial \phi_1}{\partial y_n} & \cdots & \cdots & \frac{\partial \phi_n}{\partial y_n} \end{bmatrix} \\ &= \det \left[(\pi \circ \phi)'(y) \right], \end{aligned} \tag{0.2}$$

where the second identity uses that $\det A = \det A^T$. Note that the same holds with ϕ replaced by ψ.

We now seek to justify the following equalities

$$\begin{aligned}
\int_V f(\phi(y))\, dx_1 \wedge \cdots \wedge dx_n \left(\frac{\partial \phi}{\partial y_1}, \ldots, \frac{\partial \phi}{\partial y_n} \right)\, dy \\
&= \int_V f(\phi(y)) \det \left[(\pi \circ \phi)'\,(y) \right]\, dy \\
&= \pm \int_U f(\phi(F(z))) \det \left[(\pi \circ \phi)'\,(F(z)) \right] \det[F'(z)]\, dz \\
&= \pm \int_U f(\psi(z)) \det \left[(\pi \circ \psi)'\,(z) \right]\, dz \\
&= \pm \int_U f(\psi(z))\, dx_1 \wedge \cdots \wedge dx_n \left(\frac{\partial \psi}{\partial z_1}, \ldots, \frac{\partial \psi}{\partial z_n} \right)\, dz,
\end{aligned}$$

where $\pm$ is the sign of $\det[F'(z)]$ and the last identity holds from the change of variables formula. Notice that given the reductions at the beginning, the first and last expressions here are exactly the right and left hand sides of (0.1), so once we justify these, we are done. Also note that the first and last equalities result from (0.2), and the second identity is simply the change of variables formula. We are thus left to justify the third identity here. To see this claim, first observe that $\phi(F(z)) = \phi(\phi^{-1} \circ \psi(z)) = \psi(z)$ and hence $(\pi \circ \phi)(F(z)) = (\pi \circ \psi)(z)$. Applying the chain rule to both sides of this identity, we obtain

$$(\pi \circ \phi)'(F(z))F'(z) = (\pi \circ \psi)'(z).$$

So using that $\det(AB) = \det A \det B$, the remaining identity follows. $\square$