Notes on the inverse function theorem
Math 511, Spring 2018

Theorem 0.1 (Inverse Function Theorem)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is an open set and that $F : \Omega \to \mathbb{R}^n$ is a continuously differentiable function on Ω. Suppose further that $F'(a) \in L(\mathbb{R}^n)$ is an invertible transformation for some $a \in \Omega$. Denote $b = F(a)$.*

1. *There exist open sets $U, V \subset \mathbb{R}^n$ such that $a \in U$, $b \in V$, F is one-to-one on U and $F(U) = V$. In other words the restriction of F to U, denoted $F : U \to V$ defines a bijection.*
2. *If $F^{-1} : V \to U$ denotes the inverse map of the bijection above, then F^{-1} is continuously differentiable on V and $(F^{-1})'(y) = (F'(x))^{-1}$ where $y = F(x)$.*

Proof. We first observe that if $A \in L(\mathbb{R}^n)$ is invertible and $b \in \mathbb{R}^n$ is any vector, then the mapping $x \mapsto Ax + b$ defines a bijection from $\mathbb{R}^n$ to itself with inverse given by $y \mapsto A^{-1}(y - b)$. Both mappings are the composition of a linear transformation with a translation and hence continuous. Since the inverse image of open sets under continuous maps are open, we know

$$U \text{ is open } \Leftrightarrow \{Ax + b : x \in U\} \text{ is open.}$$

Consequently, it suffices to assume that $a = b = 0$, that is $F(0) = 0$, and that $F'(0) = I$ is the identity. For if this is not the case, we can apply the theorem with these additional assumptions to $\tilde{F}(x) := (F'(a))^{-1}(F(x + a) - F(a))$, which is easily verified to be continuously differentiable on the open set $\{x \in \mathbb{R}^n : x + a \in \Omega\}$. It is then an exercise to see that the open sets U, V furnished by applying the theorem to $\tilde{F}$ yield open sets about a and b for which the conclusion of the theorem hold for F.

From now on, we assume that $a = b = 0$ so that $F(0) = 0$ and that $F'(0) = I$ with $0 \in \Omega$. Define $H : \Omega \to \mathbb{R}^n$ by $H(x) = x - F(x)$, so that $H(0) = 0$ and H is verified to be continuously differentiable with $H'(0) = 0$.

By continuity, there exists $r > 0$ such that $\overline{N_r(0)} \subset \Omega$ and $\|H'(x)\| < \frac{1}{2}$ for $x \in \overline{N_r(0)}$. Since $\overline{N_r(0)}$ is convex, Theorem 9.19 in Rudin implies that

$$|H(x) - H(x')| \leq \frac{1}{2}|x - x'|. \tag{0.1}$$

Next, recalling that $F(x) + H(x) = x$, we have that

$$\begin{aligned} |x - x'| &= |F(x) + H(x) - (F(x') + H(x'))| \\ &\leq |F(x) - F(x')| + |H(x) - H(x')| \\ &\leq |F(x) - F(x')| + \frac{1}{2}|x - x'|, \end{aligned}$$

and by rearranging the inequality, we have that

$$|x - x'| \leq 2|F(x) - F(x')|.$$

This now shows that restricting F to $\overline{N_r(0)}$ yields an injective map, for if $F(x) = F(x')$ with $x, x' \in \overline{N_r(0)}$, then $x = x'$.

Next, we have to show that F is surjective near the origin. We thus take $y \in N_{r/2}(0)$ and want to show that there exists $x \in \overline{N_r(0)}$ such that $F(x) = y$. To this end, define $G(x) := x + y - F(x) = H(x) + y$ and observe that G has a fixed point in $\overline{N_r(0)}$ if and only if $F(x) = y$ has a solution on this set:

$$x = G(x) = x + y - F(x) \Leftrightarrow 0 = y - F(x) \Leftrightarrow F(x) = y.$$

But G is easily observed to be a contraction since it is a translation of H

$$|G(x) - G(x')| = |H(x) - H(x')| \leq \frac{1}{2}|x - x'|.$$

This if we can show that $G(\overline{N_r(0)}) \subset \overline{N_r(0)}$, that is, $G : \overline{N_r(0)} \to \overline{N_r(0)}$, then G is a mapping from a complete metric space to itself, at which point the contraction mapping fixed point theorem shows that G has a unique fixed point in $\overline{N_r(0)}$. Indeed, it can be verified that a closed subset of a complete metric space defines a complete space on its own, or alternatively that any compact space is complete. Suppose $x \in \overline{N_r(0)}$, then applying (0.1) with $x' = 0$, we obtain

$$|G(x)| \leq |H(x)| + |y| < \frac{1}{2}|x| + \frac{r}{2} \leq \frac{r}{2} + \frac{r}{2} = r,$$

hence $G(\overline{N_r(0)}) \subset \overline{N_r(0)}$ and the second inequality is indeed strict since $y \in N_{r/2}(0)$. Note that the latter point implies that if $x = G(x)$, then in fact $|x| < r$, so in fact the solution to $y = F(x)$ is satisfied by some $x \in N_r(0)$.

The first part of the theorem is concluded by setting $V = N_{r/2}(0)$ and $U = N_r(0) \cap F^{-1}(N_{r/2}(0))$, which define open sets since F is continuous. Moreover, the properties established above ensure that $F : U \to V$ is a bijection.

We now prove the second half of the theorem, that $F^{-1} : V \to U$ is continuously differentiable. Note that since $\|F'(x) - I\| = \|H'(x)\| < 1/2$ on U, $F'(x)$ is invertible for $x \in U$ by Theorem 9.8(a) in Rudin. Here it is sufficient to show that if $y \in V$, then $(F^{-1})'(y)$ exists and is equal to $(F'(x))^{-1}$, where $y = F(x)$. Indeed, as soon as we establish that F^{-1} is differentiable on V, then we know that $x = F^{-1}(y)$ defines x as a continuous and function of y, so by continuity of inversion (Theorem 9.8(b)), $y \mapsto (F'(F^{-1}(y)))^{-1}$ is the composition of continuous maps, which shows that F^{-1} is continuously differentiable. Alternatively, this can be seen by using matrices: since the entries of $(F'(x))^{-1}$ are rational functions of the entries of $F'(x)$ and the partial derivatives $D_j f_i(x)$ are continuous functions, again there is continuous dependence of the entries of $(F^{-1})'(y)$ on y.

Recall that if $R(h) := F(x+h) - F(x) - F'(x)h$, then $|R(h)| = o(|h|)$ as $h \to 0$. However, here we want to define h as a function k by the relation

$$h = F^{-1}(y+k) - F^{-1}(y) = F^{-1}(y+k) - x,$$

which is a well defined injection for k such that $y + k \in V$. Hence $x + h = F^{-1}(y+k)$, equivalently,

$$F(x+h) = y + k = F(x) + k.$$

We now return the function $G = x + y - F(x)$ defined above. Recall that it is a contraction with constant $1/2$. Hence since

$$G(x+h) - G(x) = x + h + y - F(x+h) - (x + y - F(x)) = h - k$$

we have that

$$|h - k| = |G(x+h) - G(x)| \leq \frac{1}{2}|h|.$$

Hence

$$|h| \leq |k| + |h - k| \leq |k| + \frac{1}{2}|h|$$

or equivalently, $|h| \leq 2|k|$. This shows that $h \to 0$ as $k \to 0$ and that when $h \neq 0$, $\frac{1}{|k|} \leq \frac{2}{|h|}$.

We now conclude by considering

$$\begin{aligned} F^{-1}(y+k) - F^{-1}(y) - (F'(x))^{-1}k &= h - (F'(x))^{-1}k \\ &= -(F'(x))^{-1}(k - F'(x)h) \\ &= -(F'(x))^{-1}(F(x+h) - F(x) - F'(x)h). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{|F^{-1}(y+k) - F^{-1}(y) - (F'(x))^{-1}k|}{|k|} \\ &\qquad\qquad \leq 2\|(F'(x))^{-1}\| \left(\frac{|(F(x+h) - F(x) - F'(x)h)|}{|h|} \right) \end{aligned}$$

and since $h \to 0$ as $k \to 0$, the right hand side of this inequality tends to 0 as $k \to 0$. This concludes that $F^{-1}(y)$ is differentiable at y. $\square$

Theorem 0.2 (Implicit Function Theorem)**.** *Suppose $\Omega \subset \mathbb{R}^{n+m}$ is an open set and that $F : \Omega \to \mathbb{R}^n$ is continuously differentiable. Let (x, y) denote coordinates in $\mathbb{R}^{n+m}$ so that $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and write the Jacobian matrix of $F'(x, y)$ in block form as $F'(x, y) = [\frac{\partial F}{\partial x}\ \frac{\partial F}{\partial y}]$ where*

$$\frac{\partial F}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \qquad \frac{\partial F}{\partial y} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_m} \end{bmatrix},$$

so that $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ are $n \times n$ and $n \times m$ matrices respectively. If $\frac{\partial F}{\partial x}$ defines a invertible transformation in $L(\mathbb{R}^n)$ at the point (a, b), where $F(a, b) = 0$, then there exist neighborhoods V_0, W_0 with $a \in V_0 \subset \mathbb{R}^n$ and $b \in W_0 \subset \mathbb{R}^m$ and a continuously differentiable mapping $G : W_0 \to V_0$ with the property that $F(x, y) = 0$ for $(x, y) \in V_0 \times W_0$ if and only if $x = G(y)$. In other words, $F^{-1}(0) \cap V_0 \times W_0$ is the graph of G,

$$F^{-1}(0) \cap V_0 \times W_0 = \{(G(y), y) : y \in W_0\}.$$

Proof. Define $H : \Omega \to \mathbb{R}^{n+m}$ by $H(x, y) = (F(x, y), y)$ so that H is continuously differentiable and the $(n+m) \times (n+m)$ Jacobian matrix of $H'(a, b)$ in block form is

$$H'(a, b) = \begin{bmatrix} \frac{\partial F}{\partial x}(a, b) & \frac{\partial F}{\partial y}(a, b) \\ 0_{m \times n} & I_{m \times m} \end{bmatrix}, \tag{0.2}$$

where $0_{m \times n}$ is an $m \times n$ matrix of all zeros and $I_{m \times m}$ denotes the $m \times m$ identity matrix. Thus by taking determinants $\det(H'(a, b)) = \det(\frac{\partial F}{\partial x}(a, b)) \neq 0$, which shows that $H'(a,b)$ is invertible. Alternatively we can check that $H'(a,b)$ is invertible by taking any vector $(h,k) \in \mathbb{R}^{n+m}$ in the null space of $H'(a,b)$ and verifying that $(h,k) = (0,0)$. Indeed, conflating the matrices $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ with the linear transformations they define, we have

$$(0,0) = H'(a,b)(h,k) = \left(\frac{\partial F}{\partial x}(a,b)h + \frac{\partial F}{\partial y}(a,b)k, k \right),$$

and hence $k = 0$ by matching the $\mathbb{R}^m$ entries, and after inserting this into the $\mathbb{R}^n$ entries, yields $0 = \frac{\partial F}{\partial x}(a,b)h$ and hence $h = 0$.

The inverse function theorem now furnishes neighborhoods U, W with $(a,b) \in U$ and $(0,b) \in W$ such that $H : U \to W$ is bijection with continuously differentiable inverse. Shrinking U if necessary, we may assume it has the product structure $V_0 \times V_1$ where $V_0 \subset \mathbb{R}^n$, $V_1 \subset \mathbb{R}^m$ are open in their respective spaces.

We now write $H^{-1}(x,y) = (A(x,y), B(x,y))$ where $A : W \to V_0$ and $B : W \to V_1$. Hence

$$\begin{aligned} (x,y) &= H(H^{-1}(x,y)) = H(A(x,y), B(x,y)) \\ &= (F(A(x,y), B(x,y)), B(x,y)). \end{aligned}$$

Identifying both sides of the $\mathbb{R}^m$ identities here, we obtain $B(x,y) = y$ and inserting this into the $\mathbb{R}^n$ identities, we obtain that

$$x = F(A(x,y), y)$$

We now simply define $G(y) := A(0,y)$ and $W_0 := \{y \in \mathbb{R}^m : (0,y) \in W\} \cap V_1$ so that $G : W_0 \to V_0$. It is verified that W_0 is open. Thus

$$(x,y) \in F^{-1}(0) \cap V_0 \times W_0 \Leftrightarrow H(x,y) = (0,y) \text{ and } (x,y) \in V_0 \times W_0$$

and by the definitions above, the latter is equivalent to $(x,y) = H^{-1}(0,y) = (G(y), y)$ when $(x,y) \in V_0 \times W_0$. □

Theorem 0.3 (Rank Theorem)**.** *Suppose $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ are open subsets of their respective spaces and that $F : \Omega_1 \to \Omega_2$ is a continuously differentiable map. Suppose further that $F'(z)$ has constant rank for every $z \in \Omega_1$. Then given $z_0 \in \Omega_1$, there exist neighborhoods U, V of $z_0, F(z_0)$ respectively and bijections $\varphi : U \to \varphi(U)$, $\psi : V \to \psi(V)$ such that both φ, ψ and their inverses are continuously differentiable with the property that $F(U) \subset V$ and*

$$\left(\psi \circ F \circ \varphi^{-1}\right)(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$$

where in the last expression, the last $m - k$ entries vanish.

Proof. Similar to the proof of the inverse function theorem, we may assume that $z_0 = 0$ and $F(z_0) = 0$ as any needed translations can be absorbed in φ, ψ. Moreover,

$$F'(0) = \begin{bmatrix} D_1 f_1(0) & \dots & D_n f_1(0) \\ \vdots & \ddots & \vdots \\ D_1 f_m(0) & \dots & D_n f_m(0) \end{bmatrix}$$

has some $k \times k$ minor with nonvanishing determinant. By permuting coordinates in $\mathbb{R}^n$ and $\mathbb{R}^m$, we may assume that this minor is in the upper left corner, that is,

$$\det \begin{bmatrix} D_1 f_1(0) & \dots & D_k f_k(0) \\ \vdots & \ddots & \vdots \\ D_1 f_k(0) & \dots & D_k f_k(0) \end{bmatrix} \neq 0. \tag{0.3}$$

Indeed, as before, such permutations can be absorbed into φ, ψ. It is thus natural to denote coordinates $(x, y) \in \mathbb{R}^n$ where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$ and similarly $(v, w) \in \mathbb{R}^m$ where $v \in \mathbb{R}^k$, $w \in \mathbb{R}^{m-k}$. We now write

$$F(x, y) = (Q(x, y), R(x, y))$$

where $Q : \Omega \to \mathbb{R}^k$, and $R : \Omega \to \mathbb{R}^{m-k}$ are both continuously differentiable maps.

Now define $\varphi : \Omega \to \mathbb{R}^n$ by $\varphi(x, y) = (Q(x, y), y)$. Using similar notation to (0.2) as in the proof of the implicit function theorem, $\varphi'(0, 0)$ is invertible since

$$\det\big(\varphi'(0,0)\big) = \det \begin{bmatrix} \frac{\partial Q}{\partial x}(0,0) & \frac{\partial Q}{\partial y}(0,0) \\ 0_{(n-k)\times k} & I_{(n-k)\times(n-k)} \end{bmatrix} = \det\left(\frac{\partial Q}{\partial x}(0,0)\right) \neq 0.$$

since the last quantity is (0.3). The inverse function theorem now furnishes open sets $U, \tilde{U}$ containing 0 such that $\varphi : U \to \tilde{U}$ is a bijection such that φ, φ^{-1} are continuously differentiable. Shrinking $\tilde{U}$ if necessary, we may assume that it is convex.

We now write

$$\varphi^{-1}(x, y) = (A(x, y), B(x, y)) \qquad A : \tilde{U} \to \mathbb{R}^k,\ B : \tilde{U} \to \mathbb{R}^{n-k},$$

with A, B continuously differentiable. Observe that by the definition of φ,

$$(x, y) = \varphi(A(x, y), B(x, y)) = \Big(Q\big(A(x, y), B(x, y)\big), B(x, y)\Big).$$

Thus by matching entries, we have that $B(x,y) = y$ which implies that

$$\varphi^{-1}(x,y) = (A(x,y),y) \text{ and } x = Q(A(x,y),y).$$

We now may write for some $\mathbb{R}^{m-k}$-valued function $\tilde{R}$, continuously differentiable on $\tilde{U}$

$$F \circ \varphi^{-1}(x,y) = F(A(x,y),y) = (Q(A(x,y),y), R(A(x,y),y)) = (x, \tilde{R}(x,y)).$$

Hence

$$(F \circ \varphi^{-1})'(x,y) = \begin{bmatrix} I_{k\times k} & 0_{k\times(n-k)} \\ \frac{\partial \tilde{R}}{\partial x}(x,y) & \frac{\partial \tilde{R}}{\partial y}(x,y) \end{bmatrix}. \tag{0.4}$$

But by the chain rule, for $(x,y) \in \tilde{U}$, $(F \circ \varphi^{-1})'(x,y)$ has rank k. Indeed,

$$(F \circ \varphi^{-1})'(x,y) = F'(\varphi^{-1}(x,y))(\varphi^{-1})'(x,y),$$

and since $F'(z)$ has constant rank with $(\varphi^{-1})'(x,y)$ is invertible, this is the composition of a rank k map with an invertible map, which yields a rank k linear mapping. But given (0.4), the first k columns of $(F \circ \varphi^{-1})'(x,y)$ are linearly independent, which means that $\frac{\partial \tilde{R}}{\partial y} = 0$ on $\tilde{U}$ since otherwise, the matrix would have rank larger than k. But since $\tilde{U}$ is convex, we have that $\tilde{R}$ is independent of y (cf. Theorem 9.19 and its corollary), that is $\tilde{R}(x,y) = S(x)$ for some continuously differentiable function S defined on the open set

$$\tilde{V} := \{x \in \mathbb{R}^k : (x,0) \in \tilde{U}\}.$$

This now shows that $(F \circ \varphi^{-1})(x,y) = (x, S(x))$.

We finally define for $v \in \tilde{V}$, $w \in \mathbb{R}^{m-k}$, $\psi(v,w) = (v, w - S(v))$. This defines a continuously differentiable bijection with explicit inverse $\psi^{-1}(s,t) = (s, t + S(s))$, which satisfies

$$(\psi \circ F \circ \varphi^{-1})(x,y) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0),$$

where the latter entries in the last two expressions are in $\mathbb{R}^{m-k}$. Defining V to be the open set $V := \{(v,w) \in \mathbb{R}^m : v \in \tilde{V}\}$, the proof is now concluded. □