Strichartz Estimates for the Schrödinger Equation in Exterior Domains

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Joint work with:

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- Christopher Sogge (Johns Hopkins University)

The Schrödinger equation on $\mathbb{R}^n$

- Initial value problem for the Schrödinger equation

$$(i\partial_t + \Delta)u(t,x) = 0, \qquad u(0,x) = f(x), \qquad u(t,x) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$$

- Two fundamental properties

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|u(0,\cdot)\|_{L^2(\mathbb{R}^n)} \qquad \text{(mass conservation)}$$
$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq c_n t^{-\frac{n}{2}} \|u(0,\cdot)\|_{L^1(\mathbb{R}^n)} \qquad \text{(dispersive inequality)}$$

- Together, they yield Strichartz estimates

$$\|u\|_{L^p(\mathbb{R};L^q(\mathbb{R}^n))} \leq C\|f\|_{L^2(\mathbb{R}^n)}, \qquad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad p, q > 2$$

Strichartz estimates with Sobolev regularity

$$\|u\|_{L^p(\mathbb{R};L^q(\mathbb{R}^n))} \leq C\|f\|_{H^s(\mathbb{R}^n)}, \qquad \frac{2}{p} + \frac{n}{q} \geq \frac{n}{2} - s \; (*)$$

- When equality holds in $(*)$, the estimate is *scale-invariant*. Otherwise, there is a *loss of derivatives*.
- Can combine estimates when $s = 0$ with Sobolev embedding to get general case with $s > 0$
- When $s > 0$, estimates are *subcritical*, they do not use the full rate of dispersion

Frequency localized estimates

- Direct method: Littlewood-Paley decomposition

$$u = \sum_{\lambda = 2^k} u_\lambda, \qquad \text{supp}(\widehat{u}_\lambda(t,\cdot)) \subset \{|\xi| \approx \lambda\},$$

- Standard squarefunction estimate reduces Strichartz to

$$\|u_\lambda\|_{L^p(L^q)} \leq C\lambda^s \|u_\lambda(0,\cdot)\|_{L^2}$$

- Kernel of the solution map at frequency λ on satisfies a refined dispersive inequality

$$K_\lambda(t,x,y) = \int e^{i(x-y)\cdot\xi - it|\xi|^2} \beta(\lambda^{-1}\xi)\, d\xi$$

$$|K_\lambda(t,x,y)| \leq C \min(\lambda^n, t^{-\frac{n}{2}}) \approx C(\lambda^{-2}+t)^{-\frac{n}{2}} \leq \lambda^{2\alpha} t^{-\frac{n}{2}+\alpha}$$

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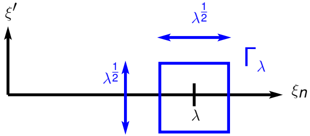
Knapp example

- High frequency Knapp example: solve eqn. w/ $\hat{f}$ the characteristic fcn of $\Gamma_\lambda = \{(\xi', \xi_n) : |\xi_n - \lambda| \leq \lambda^{\frac{1}{2}}, |\xi'| \leq \lambda^{\frac{1}{2}}\}$

$$v_\lambda(t,x) = \int_{\Gamma_\lambda} e^{ix\cdot\xi - it|\xi|^2} d\xi,$$



- Linearize the phase $\Rightarrow |v_\lambda(t,x)| \approx \lambda^{\frac{n}{2}}$ over the set $\{|x'| \ll \lambda^{-\frac{1}{2}}, |t| \ll \lambda^{-1}, |x_n - 2t\lambda| \ll \lambda^{-\frac{1}{2}}\}$

Knapp example, continued

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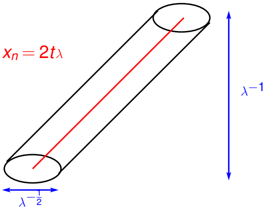

- Computing the ratio forces $\frac{2}{p} + \frac{n}{q} \leq \frac{n}{2}$

$$\|v_\lambda\|_{L^p_t(L^q_x)} / \|v_\lambda(0,\cdot)\|_{H^s} \geq c\lambda^{\frac{1}{2}(\frac{2}{p}+\frac{n}{q}-\frac{n}{2})}, \qquad \lambda \to \infty$$

- Strict inequality (equiv. $s > 0$) gives exponents which are *subcritical*

Schrödinger equations on Riemannian manifolds

- Staffilani-Tataru, Burq-Gérard-Tzvetkov: local (small time) parametrix constructions
- Take a Littlewood-Paley decomposition, consider $u_\lambda(t,\cdot)$ spectrally localized to frequencies $\approx \lambda = 2^k \geq 1$
- Speed of propagation is finite, but proportional to λ
- A parametrix which inverts the eqn locally (in space) will have bounded error over time intervals of size λ^{-1}
- A priori, this generates a loss of $\frac{1}{p}$ derivatives ($|I_\lambda| \approx \lambda^{-1}$)

$$\|u_\lambda\|_{L^p([0,T];L^q)} = \Big(\sum_{I_\lambda \in [0,T]} \|u_\lambda\|^p_{L^p(I_\lambda;L^q)} \Big)^{\frac{1}{p}} \leq C_T \lambda^{s+\frac{1}{p}} \|u_\lambda(0,\cdot)\|_{L^2}$$

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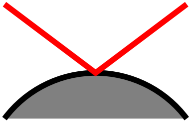
Obstacle Problems

- Let $\Omega = \mathbb{R}^n \setminus \mathcal{K}$ be a domain in $\mathbb{R}^n$ exterior to a compact obstacle $\mathcal{K}$ with smooth boundary
- Consider the initial value problem, with homogeneous BC

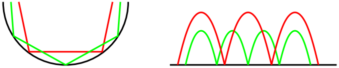
$$u(t,\cdot)|_{\partial\Omega} = 0 \text{ (Dirichlet)} \qquad \text{or} \qquad \frac{\partial u}{\partial \nu}(t,\cdot)\Big|_{\partial\Omega} = 0 \text{ (Neumann)}$$

- The local and global structure of the boundary can affect the flow of energy and dispersion
- Energy propagates along *broken* bicharacteristics



Gliding Rays

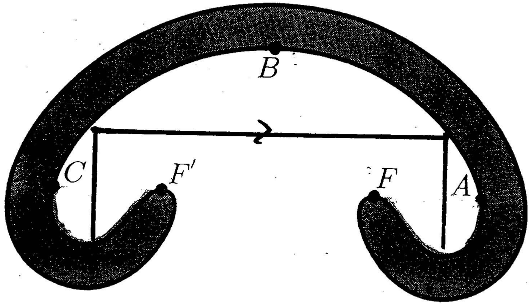
- "Local" complications: points of convexity in $\partial\Omega$



- Broken rays reflect in the boundary several times, complicating parametrix constructions
- Two works of Ivanovici:
  - Whispering gallery modes provide a counterexample for a range of exponents including the critical case ($\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$)
  - Strichartz estimates hold for domains with strictly concave boundary (Melrose-Taylor parametrix)

Trapped Rays

- “Global" issues: obstacle may create trapped rays



- Elliptic trapping $\rightarrow$ no hope for scale-inv. estimates
- Hyperbolic trapping $\rightarrow$ some hope

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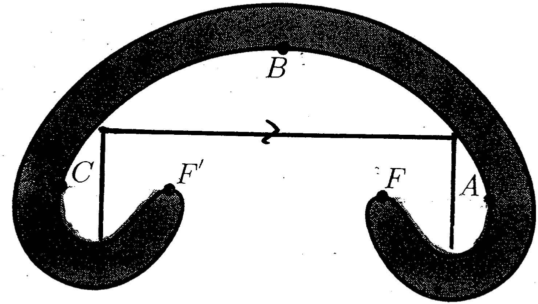

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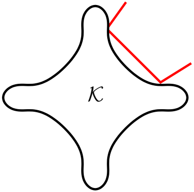
Non-trapping assumption

- From here on, we work with a *non-trapping* assumption: every unit speed broken bicharacteristic escapes a compact subset in finite time
- e.g. star-shaped obstacle



Local Smoothing Estimates

- Burq-Gérard-Tzvetkov: For non-trapping obstacles

$$\|\psi u\|_{L^2([0,T];H^{s+\frac{1}{2}}(\Omega))} \leq C\|f\|_{H^s(\Omega)}, \qquad \psi \in C_c^\infty(\overline{\Omega}) \qquad \text{(LS)}.$$

- Analogous bounds in $\mathbb{R}^n$ due to Kato, Constantin-Saut, Sjölin, Vega, and others
- Wave packet at frequency λ should spend time $\approx \frac{1}{\lambda}$ in supp(ψ), taking L^2 in time should yield a gain of $\sqrt{\frac{1}{\lambda}}$
- LS reduces or eliminates losses that come from working locally
  - Staffilani-Tataru: non-trapping metric perturbations of Δ
  - Burq et. al., Anton: estimates with a loss in domains

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Using local smoothing estimates

$$\|\psi u\|_{L^2([0,T];H^{s+\frac{1}{2}}(\Omega))} \leq C\|f\|_{H^s(\Omega)}, \;\psi \in C_c^\infty(\overline{\Omega}).$$

- Deals with error terms that arise in localizing near $\partial\Omega$
- Consider $u_\lambda(t,\cdot)$ within a chart U which flattens $\partial\Omega$
- Take a space-time decomposition of the solution into sets $I_\lambda \times U$, $|I_\lambda| \approx \frac{1}{\lambda}$ is a time interval
- When $\operatorname{dist}(I_\lambda, J_\lambda) \geq \frac{C}{\lambda}$, solution over $I_\lambda \times U$ should have almost no influence on the solution over $J_\lambda \times U$
- Independence of solution over these time intervals $\Rightarrow$ estimates with no loss of derivatives

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Small-time estimates

- Bottom line: Matters are reduced to establishing local/small-time (or semiclassical) estimates
- For $u_\lambda(t,\cdot)$ concentrated in a chart U, $t \in [0,\lambda^{-1}]$

$$\|u_\lambda\|_{L^p([0,\lambda^{-1}];L^q)} \leq C\lambda^s \|u_\lambda(0,\cdot)\|_{L^2}$$

- Anton, B.-Smith-Sogge: Estimates with a loss
- No loss estimates for Dirichlet BC's
  - Planchon-Vega: Bilinear Virial identities which give estimates for $p = q = 4$
  - Ivanovici: strictly concave boundary

Our Result

Theorem (B.-Smith-Sogge): The scale-invariant Strichartz estimates

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C\|f\|_{H^s(\Omega)}, \qquad \frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s,$$

hold for solutions in non-trapping exterior domains $\mathbb{R}^n \setminus \mathcal{K}$, provided

$$\begin{cases} \frac{3}{p} + \frac{n}{q} \leq \frac{n}{2} & n \leq 3, \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} & n \geq 4 \end{cases}$$

- For compact domains, we have a loss of $\frac{1}{p}$ derivatives, (cp. no boundaries: Burq-Gérard-Tzvetkov, Staffilani-Tataru)

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C\|f\|_{H^{s+\frac{1}{p}}(\Omega)}$$

The parametrix

Adapts a construction for the wave eqn. due to Smith-Sogge

- Work in suitable coordinates that flatten the boundary, get a variable coefficient problem
- Reflect the coefficients and the solution in the boundary
- Yields a PDE in an open set in $\mathbb{R}^n$, but with rough coefficients
- Now perform a Littlewood-Paley decomposition of the solution in frequency $u = u_0 + \sum_{\lambda=2^k} u_\lambda$

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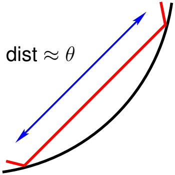
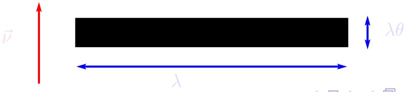
- Speed λ rays reflecting at an angle θ should not return until a time $t_\theta \approx \lambda^{-1}\theta$



- For each $\theta = 2^{-j} \in [\lambda^{-1/3}, 1]$, localize solution in frequency again to sets

$$\text{supp}(\hat{u}_{\lambda,\theta}(t,\cdot)) \subset \{|\xi| \approx \lambda,\ |\langle \vec{\nu}, \xi/|\xi|\rangle| \approx \theta\} \qquad (\vec{\nu} = \text{unit normal})$$



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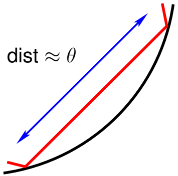

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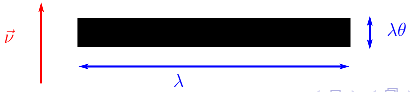

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- This yields Strichartz estimates over small slabs in space-time $I_\theta \times U$, $|I_\theta| \approx \lambda^{-1}\theta$
- Loss of $\theta^{-\frac{1}{p}}$ derivatives can be countered by gains in the dispersive estimates

$$|K_{\lambda,\theta}(t,x,y)| \leq C(\lambda^{-2}+t)^{-\frac{n-1}{2}}(\lambda^{-2}\theta^{-2}+t)^{-\frac{1}{2}}$$

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Energy critical equations in 3+1 dim.

- Semilinear Schrödinger equation for Dirichlet BC:

$$(i\partial_t + \Delta)v = \pm|v|^4 v, \qquad v(t,\cdot)|_{\partial\Omega} = 0$$
$$v(0,x) = g(x) \in H^1(\Omega)$$

- Conservation Law: The following is conserved

$$E(v) = \int_\Omega \frac{1}{2}|\nabla_x v(t,x)|^2 \mp \frac{1}{6}|v(t,x)|^6 \, dx$$

- $H^1(\Omega) \hookrightarrow L^6(\Omega)$, but the two spaces scale the same way, places a premium on scale-invariant estimates

Energy critical equations

- We recover a recent result of Ivanovici-Planchon:
- **Theorem** The energy critical equation is locally well-posed on non-trapping exterior domains $\Omega = \mathbb{R}^3 \setminus \mathcal{K}$. If $\|g\|_{H^1(\Omega)} \leq \epsilon$, the solution exists globally in time
- Formally, we have an $L^4_t L^\infty_x$ estimate in $n = 3$

$$\|u\|_{L^4([-T,T];L^\infty(\Omega))} \leq C_T \|f\|_{H^1(\Omega)}$$

- Allows for iteration to a fixed point

$$\|\nabla(|v|^4 v)\|_{L^1(L^2)} \leq C\|v\|^4_{L^4(L^\infty)} \|\nabla v\|_{L^\infty(L^2)}$$