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Nonrelativistic Kapitza-Dirac scattering

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We use techniques of singular perturbation theory to investigate the scattering of nonrelativistic charged particles by a standing light wave (Kapitza-Dirac scattering). Unlike previous treatments, we give explicit results for the effects of the time-dependent part of the field. For low field intensity or low particle energy, we show that the leading-order effects can be found from an averaged equation, and we compute corrections. For the strong fields that can be produced by modern lasers and/or high particle energies, we show that the time dependence of the potential leads to focusing. Our methods can be applied to other problems with time-periodic potentials.

I. INTRODUCTION

In 1933 Kapitza and Dirac[1] predicted that electrons traveling in a standing light wave would be reflected from the planes of peak intensity. The probability per electron for reflection was proportional to the square of the product of the field intensity and the interaction time. Because the light sources available at that time were too weak to generate an observable effect, the subject was neglected until the discovery of the laser. Since then, numerous theoretical and experimental studies have been devoted to this subject.

The earliest of these theoretical papers[2,3] followed closely the original treatment of Kapitza and Dirac inasmuch as they relied on a first-order expansion of the wave function in terms of the standing-wave field. Like the original paper, they are not valid for presently available laser intensities since they would predict scattering probabilities in excess of unity. Fedorov[4] published the first extensive treatment of the problem. By neglecting the time-dependent part of the standing wave, he was able to rewrite the Schrödinger equation in the form of a Mathieu equation. Solutions were then found for the cases of a low intensity field and a high intensity field. Unfortunately, neither of these cases corresponded to intensities used in the experiments.

Gush and Gush[5] used the nonrelativistic Green's function for an electron in a standing-wave field to produce an exact solution to the problem when the time-dependent part of the field is neglected. This treatment is not only valid for all intensities up to the point where the time-dependent part of the field becomes important but also for electron momenta that do not satisfy the Bragg conditions. Furthermore, the probabilities for higher-order reflections are treated. Unfortunately, the final expressions for the scattering probabilities are not in terms of known functions and are unwieldy.

The question of what role, if any, the time-dependent part of the standing-wave field plays in the scattering process is still an unresolved issue. Gush and Gush[5] and others have argued that the time-dependent portion of the field can safely be neglected for the intensities used in the reported experiments. This conclusion has been disputed by Ehlotzky *et al.*[6] whose calculations show that the time dependence of the field has a significant influence on the scattering of the electron. The validity of their calculation remains in doubt in view of the fact that except for small interaction times and intensities the scattering probabilities for the time-averaged case differ from those predicted by the more exact treatment of Gush and Gush.[5]

In this paper, the role of the time-dependent portion of the standing wave is investigated using several different approximation techniques. The geometry and governing equations are introduced in Sec. II. Various nondimensional parameters are also defined in this section and interpreted in terms of physical quantities. In Sec. III multiple-time-scale perturbation theory is used to calculate scattering probabilities for the case when the standing-wave field is not too strong. This case corresponds to the one previously treated in the literature. The time-averaged equation is derived with no further assumption and the limits of its validity are discussed. This equation is then solved in terms of Mathieu functions and several new features of the scattering probabilities are discussed. In Sec. IV we treat the case that is characterized by strong coupling and high electron energies. This case is discussed using a semiclassical approximation. The wave function is again found to be quasiperiodic within the limitations of perturbation theory. Under appropriate conditions the corresponding classical problem leads to focusing. In quantum-mechanical terms the focusing is exhibited as sharp localized maxima in the scattering probabilities. In Sec. V the conclusions are drawn.

II. MODEL

In this paper we will consider the problem of a nonrelativistic, quantum-mechanical electron interacting with a

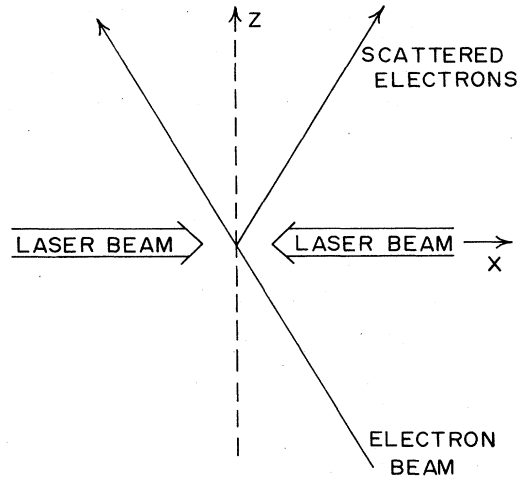

FIG. 1. A schematic of Kapitza-Dirac scattering.

classical, electromagnetic, standing wave as illustrated in Fig. 1. The electron has initial energy E and momentum $\mathbf{p}$ which is chosen so as to lie in the $\mathbf{x}'$-$\mathbf{z}'$ plane. A standing wave of frequency ω lies along the $\mathbf{x}'$ axis and is assumed to be plane polarized along the $\mathbf{y}'$ axis. The interaction is turned on at $t=0$ and continues for a time $T=L/v$ where $v=|\mathbf{p}|/m$ is the electron velocity, m is the electron mass, and L is the interaction length.

The standing wave is given by the vector potential,

$$\mathbf{A}=A[\cos(kx'-\omega t)+\cos(kx'+\omega t)]\hat{y}$$
$$=2A\cos(kx')\cos(\omega t)\hat{\mathbf{y}}\ . \tag{2.1}$$

Here A is the intensity of one of the two counterpropagating fields that combine to make the standing wave and $k=\omega/c$.

Because the electron is nonrelativistic and spin effects are not important for most cases of interest, its behavior is described by the solution of the Schrödinger equation with the external field (2.1). This equation can be written in the following conventional nondimensional form:

$$\left[\frac{\partial^2}{\partial x^2}-2q\cos(2x)[1+\cos(2\tau/\epsilon)]\right]\psi=-i\frac{\partial\psi}{\partial\tau} \tag{2.2a}$$

where

$$x=kx'\ , \tag{2.2b}$$

$$\tau=\frac{\hbar\omega^2 t}{2mc^2}\ , \tag{2.2c}$$

$$q=\frac{1}{2}\frac{e^2}{\hbar\omega}A^2\ , \tag{2.2d}$$

$$\epsilon=\frac{\hbar\omega}{2mc^2}\ . \tag{2.2e}$$

For all cases in which an experiment is feasible $\hbar\omega\ll mc^2$ so that

$$\epsilon\ll 1\ . \tag{2.3}$$

The initial condition for the wave function is specified by its free-space value at the time when the interaction is turned on

$$\psi(x,\tau=0)=\exp(i\beta x) \tag{2.4}$$

where

$$\beta=p_x/\hbar k\ . \tag{2.5}$$

A cursory inspection of Eq. (2.2a) shows that two types of interaction of the electron with the field can occur. In one case the particle only exchanges momentum with the field. This is the elastic scattering mode that has been extensively investigated.[1–5] The term in the potential that only has a spatial dependence determines this interaction. The remaining term is a product of a spatial and temporal piece. Here the electron experiences a change in both momentum and energy each time it interacts with the field. This is the inelastic portion of the scattering.

Because the potential in Eq. (2.2a) only permits the electron to change its nondimensional momentum by multiples of 2, the final state of the electron must be characterized by a momentum p_n which satisfies

$$p_n=\hbar k\beta_n=\hbar k(\beta+2n),\quad n=0,\pm1,\pm2,\ldots\ . \tag{2.6}$$

The probability amplitude that after an interaction time τ_f the electron is in a state with momentum p_n is given by

$$P_n(\beta,\tau_f)=\left|\frac{1}{\pi}\int_0^{\pi}\exp(-i\beta_n z)\psi(z,\tau_f)dz\right|^2\ . \tag{2.7}$$

These probabilities must satisfy the condition

$$\sum_{n=-\infty}^{\infty}P_n=1\ . \tag{2.8}$$

The strength of the interaction is determined by the parameter q introduced in Eq. (2.2d). In order to provide a physical interpretation for this quantity it is convenient to rewrite it in terms of the individual wave intensities $I=ck^2A^2/8\pi$ so that

$$q=\left[\frac{2\pi r_0\lambda\!\!\!^{-2}I}{\hbar\omega c}\right]\frac{2mc^2}{\hbar\omega}\ , \tag{2.9}$$

where r_0 is the classical electron radius and $\lambda\!\!\!^{-}=1/k$. When I is rewritten in terms of the photon number density ρ ($I=\hbar\omega c\rho$) then the quantity in the square brackets reduces to $2\pi r_0\lambda\!\!\!^{-2}\rho$. The coupling constant q is therefore proportional to the number of photons in a cylinder of length $2r_0$ and radius $\lambda\!\!\!^{-}$. For available laser sources q can reach values of $\sim 10^8$ in unfocused beams although $q\lesssim 10$ is more characteristic of the published experiments.[7–12]

Because we have been unable to find an exact solution for Eq. (2.2a), we have had to resort to various perturbation schemes to find approximate expressions for the wave function in different regimes. These regimes are characterized by the relative size of the adjustable parameters q, ϵ, and β. In the remainder of this paper we will be concerned with the range of values of the parameters accessible to the experimentalist.

III. WEAK COUPLING

The weak-coupling limit is characterized by the two conditions

$$1/\epsilon \gg q$$

and

$$1/\epsilon \gg |\beta| \ .$$

This case covers the conditions that characterize previous experiments,[7–12] i.e., $|\beta| \leq 10$, $q \leq 10$, and $\epsilon^{-1} \approx 10^6$. Under these conditions multiple-time-scale (MTS) perturbation theory[13] can be used to find an approximate expression for the wave function.

In order to apply this perturbation analysis, the wave function ψ is assumed to have the following asymptotic expansion:

$$\psi(z,\tau) = \sum_{n=0}^{N} \epsilon^n \psi_n(x,t) + O(\epsilon^{N+1}) \ , \tag{3.1}$$

where each ψ_n is a function of the time scales

$$t_k = \epsilon^k \tau, \quad k = -1,0,1,\ldots \ . \tag{3.2}$$

The reason for considering a sequence of time scales is that each mode of the wave function has a characteristic time scale $\tilde{t} = \omega(\epsilon)\tau$ that depends on ϵ. The most convenient means of studying all the modes simultaneously is therefore to isolate effects to a given order of ϵ.

For this problem we need to expand the wave function to order ϵ^2 so that only four time scales are necessary, i.e., t_{-1}, t_0, t_1, t_2. Of these four time scales, t_1 will make no contribution to the calculation so it will be eliminated from the start. The time derivative in Eq. (2.2a) is then written as

$$\frac{\partial}{\partial \tau} = \epsilon^{-1} \frac{\partial}{\partial t_{-1}} + \frac{\partial}{\partial t_0} + \epsilon^2 \frac{\partial}{\partial t_2} \ . \tag{3.3}$$

Once Eqs. (3.1)–(3.3) are substituted in Eq. (2.2a) and coefficients of the various powers of ϵ are set equal to zero, the following set of coupled differential equations emerges:

$$O(\epsilon^{-1}): \ \frac{\partial \psi_0}{\partial t_{-1}} = 0 \ , \tag{3.4}$$

$$O(\epsilon^0): \ L\psi_0 = -i \frac{\partial \psi_1}{\partial t_{-1}} \ , \tag{3.5}$$

$$O(\epsilon^1): \ L\psi_1 = -i \frac{\partial \psi_2}{\partial t_{-1}} \ , \tag{3.6}$$

$$O(\epsilon^2): \ L\psi_2 + i \frac{\partial \psi_0}{\partial t_2} = -i \frac{\partial \psi_3}{\partial t_{-1}} \ , \tag{3.7}$$

where

$$L \equiv \frac{\partial^2}{\partial x^2} - 2q \cos(2x)[1 + \cos(2t_{-1})] + i \frac{\partial}{\partial t_0} \ . \tag{3.8}$$

The initial conditions are then given by

$$\psi_n(x, t_{-1} = 0, t_0 = 0, t_2 = 0) = \delta_{n0} \exp(i\beta x) \ . \tag{3.9}$$

Equation (3.4) requires that ψ_0 is independent of t_{-1}, i.e., $\psi_0 = \psi_0(x, t_0, t_2)$. With this result Eq. (3.5) can be integrated over t_{-1} to give an expression for ψ_1 in terms of ψ_0,

$$\psi_1 = i[(\bar{L}\psi_0)t_{-1} - q \cos(2x)\sin(2t_{-1})\psi_0] \ . \tag{3.10}$$

In general, this equation would contain an integration constant that is a function of x, t_0, and t_2. Because this function can be shown to be a part of ψ_0, it will not appear in this discussion. The secular term in Eq. (3.10) is removed by requiring that the coefficient of t_{-1} be set equal to zero. This condition yields

$$\bar{L}\psi_0 \equiv \left[\frac{\partial^2}{\partial z^2} - 2q \cos(2x) + i \frac{\partial}{\partial t_0} \right] \psi_0 = 0 \tag{3.11}$$

and leaves

$$\psi_1 = -iq \cos(2x)\sin(2t_{-1})\psi_0 \ . \tag{3.12}$$

Equation (3.11) is the one we would have obtain if we had followed the traditional approach and averaged Eq. (2.2a) over the rapid oscillations.

The solution of Eq. (3.11) which satisfies the initial condition (3.9) is

$$\psi_0(x,t_0,t_2) = \sum_{n=-\infty}^{\infty} \alpha_n(t_2) \mathrm{me}_{\beta+2n}(x,q) \exp[-i\lambda_{\beta+2n}(q)t] \tag{3.13}$$

with

$$\alpha_n(0) = C_{-2n}^{\beta+2n}(q) \ . \tag{3.14}$$

Here $\mathrm{me}_\alpha(z,q)$ is a Mathieu function[14] of order α and $\lambda_\alpha(q)$ is its eigenvalue. The constants $C_{2r}^{\alpha}(q)$ are the Fourier coefficients of $\mathrm{me}_\alpha(z,q)$, i.e.,

$$\mathrm{me}_\alpha(z,q) = \sum_{r=-\infty}^{\infty} C_{2r}^{\alpha}(q) \exp[i(\alpha + 2r)z] \ . \tag{3.15}$$

The substitution of Eq. (3.12) into Eq. (3.6) and the subsequent integration over t_{-1} yields the following expression for ψ_2:

$$\begin{aligned} \psi_2(x,t_{-1},t_0,t_2) &= 2q[\cos(2t_{-1}) - 1]\cos(2x)\psi_0 + \sin(2x)\frac{\partial \psi_0}{\partial x} \\ &\quad + \frac{q^2}{4}\cos(2x)[\cos(4t_{-1}) - 1]\psi_0 + B(x,t_0,t_2) \ , \end{aligned} \tag{3.16}$$

where B is an undetermined function that satisfies the initial condition

$$B(x,0,0) = 0 \ . \tag{3.17}$$

There are no secular terms this time. In a similar manner Eq. (3.7) can be integrated over t_{-1} after the substitution of Eq. (3.16). Because we are solving for the wave function to $O(\epsilon^2)$, we only need to investigate the differential equation,

$$\left[\frac{\partial^2}{\partial z^2}-2q\cos(2x)+i\frac{\partial}{\partial t_0}\right]B = -i\frac{\partial\psi_0}{\partial t_2}+8q\left[\cos(2x)\psi_0+2\sin(2x)\frac{\partial\psi_0}{\partial x}-\cos(2x)\frac{\partial^2\psi_0}{\partial x^2}\right] +3q^2[1-\cos(4x)]\psi_0 \tag{3.18}$$

that arises from setting the coefficient of t_{-1} to zero. The left-hand side of Eq. (3.18) contains the same operator that occurs in Eq. (3.11). We can therefore write B as

$$B(x,t_0,t_2)=\sum_m b_m(x,t_2)\exp[-i\lambda_{\beta+2m}(q)t_0] . \tag{3.19}$$

When Eqs. (3.13) and (3.19) are substituted in Eq. (3.18) and the coefficients of $\exp[i\lambda_{\beta+2m}(q)t_0]$ are set equal to zero, the following equation arises:

$$\left[\frac{\partial^2}{\partial x^2}-2q\cos(2x)+\lambda_{\beta+2n}\right]b_n=-i\frac{\partial\alpha_n}{\partial t_2}+8\alpha_n q\left[\cos(2x)+2\sin(2x)\frac{\partial}{\partial x}+\cos(2x)\lambda_{\beta+n}\right] -q^2[5+11\cos(4x)]\mathrm{me}_{\beta+2n}(x,q) . \tag{3.20}$$

We can now apply Fredholm's alternative theorem[15] to this equation which requires that

$$i\frac{\partial\alpha_n}{\partial t_2}=\gamma_n\alpha_n , \tag{3.21}$$

where

$$\gamma_n(q)=\frac{1}{\pi}\int_0^{\pi}\mathrm{me}^*_{\beta+2n}(x,q)\hat{\Theta}(q,x)\mathrm{me}_{\beta+2n}(x,q)dx \tag{3.22}$$

and

$$\hat{\Theta}(q,x)=8q\left[[1+\lambda_{\beta+2n}(q)]\cos(2x)+2\sin(2x)\frac{\partial}{\partial x}\right]-q^2[5+11\cos(4x)] . \tag{3.23}$$

Equation (3.21) with initial condition (3.14) has the obvious solution,

$$\alpha_n(q)=C_{-2n}^{\beta+2n}(q)\exp[-i\gamma_n(q)t_2] . \tag{3.24}$$

Equations (3.12), (3.13), and (3.24) can be combined by using Eqs. (3.1) and (3.2) to give an approximate expression for the wave function

$$\psi(x,t)\simeq[1-i\epsilon q\sin(2\tau/\epsilon)]\sum_{n=-\infty}^{\infty}C_{-2n}^{\beta+2n}(q)\mathrm{me}_{\beta+2n}(q,x)\exp\{-i[\lambda_{\beta+2n}(q)+\epsilon^2\gamma_n(q)]\tau\} . \tag{3.25}$$

This wave function is still properly normalized to terms of $O(\epsilon^2)$. For the condition that we assumed at the beginning of this section ($\epsilon^{-1}\gg q$ and $\epsilon^{-1}\gg|\beta|$) the corrections to the lowest-order wave function are negligible. As an example consider the terms in the first set of square brackets. If we calculate the probability (2.7) that the electron will be in a particular momentum state after a time τ then the second term in the square bracket makes a contribution that is a factor $(\epsilon q)^2$ smaller than the first and can therefore be safely ignored as long as $\epsilon^{-1}\gg q$. Even if there were some means of detecting this small amplitude correction, we would have to contend with the fact that this term is oscillating at the frequency of the standing-wave field.

The second term in the argument of the exponential in Eq. (3.25) also does not appreciably influence the scattering probability as long as we are considering reasonable interaction times ($\tau\lesssim 10$). This term introduces a slow modulation in time to the wave function. For $q\ll\epsilon^{-1}$, the period of oscillation is much greater than conceivable interaction times so again the correction induced by the oscillating portion of the potential can be neglected. We are thus led to the conclusion that as long as $q\ll\epsilon^{-1}$ and $|\beta|\ll\epsilon^{-1}$ the temporally oscillating term in the vector potential can be safely neglected. This result contradicts the conclusions of Ref. 6.

The perturbation calculation is no longer valid when $\epsilon q\simeq 1$ since the second term in the first set of square brackets of Eq. (3.25) becomes comparable with the first term. If we return to the definition of q, Eq. (2.9), and ϵ, Eq. (2.2e), then the product can be rewritten as

$$\epsilon q=\frac{2\pi r_0\lambda^2 I}{\hbar\omega c}=2\pi r_0\lambda^2\rho \tag{3.26}$$

where $I=\hbar\omega c\rho$ has been used in the last step and ρ is the photon number density. An estimate of when the time variation of the vector potential becomes important is $2\pi r_0\lambda^2\rho=1$. In other words, there is one photon in the vicinity of the electron at all times. This condition makes

sense. The time-dependent part of the standing wave represents a stimulated emission or absorption of two photons by an electron. This process will not be probable unless there are two photons in the vicinity of the electron. Because a photon cannot be localized perpendicular to the direction of motion to a distance less than a wavelength, the vicinity of an electron is the volume $2\pi r_0\lambda^2$. We have therefore returned to the condition given by Eq. (3.26). For $\lambda=1$ μm, Eq. (3.26) requires $I\simeq 10^{14}$ W/cm. Although these intensities can be achieved for focused laser beams they cannot be achieved in the 1-cm beams needed to generate a reasonable interaction time for the electrons. On the other hand, CO_2 lasers ($\lambda=10$ μm) require $I\simeq 10^{11}$ W/cm^2 for Eq. (3.16) to be satisfied. This is presently possible.

When the small terms in Eq. (3.25) are dropped, the wave function assumes the form

$$\psi(x,t)=\sum_{n=-\infty}^{\infty} C_{-2n}^{\beta+2n}(q)\mathrm{me}_{\beta+2n}(x,q) \times\exp[-i\lambda_{\beta+2n}(q)\tau]\ . \tag{3.27}$$

The probability for scattering from an initial state with momentum $p_i=\beta\hbar k$ to a final state with momentum $p_f=(\beta+2r)\hbar k$ after a time τ is then given by Eq. (2.7) as

$$P_r(\beta,\tau)=\left|\sum_{n=-\infty}^{\infty} C_{-2n}^{\beta+2n}(q)C_{2(r-n)}^{\beta+2n}(q) \times\exp[-i\lambda_{\beta+2n}(q)\tau]\right|^2 \tag{3.28}$$

and satisfies Eq. (2.8). Although very different in form, the continued-fraction expression for the scattering amplitude derived in Ref. 5 is identical to Eq. (3.27). The advantages of the Mathieu function expansion over its continued fraction form is that Eq. (3.28) is easier to approximate analytically and evaluate numerically. Because the Fourier coefficients of the Mathieu functions satisfy a three-term recursion relation,[14] Mathieu's equation can be written as a matrix eigenvalue problem where the matrix is tridiagonal. The eigenvalues $\lambda_{\beta+2n}$ and the Fourier coefficients $c_{2r}^{\beta+2n}$ are then easily found numerically using any standard program to diagonalize the tridiagonal matrix.

Figure 2 shows the scattering probability as a function of q for $p_i=\hbar k$; $p_f=\hbar k,-\hbar k,3\hbar k,-3\hbar k$ and $\tau=2$ for a standing wave with a wavelength of 1 μm. The interaction time is chosen so as to represent the approximate time a 200-eV electron takes to travel 1 cm. These values of the parameters are chosen as representative of a feasible experiment. One noticeable feature of these graphs is the increasing sensitivity of the scattering probability to the standing-wave intensity. This result is not surprising since increasing q increases the number of terms that contribute to the summation in Eq. (3.13). Each additional term adds another and higher frequency component to the scattering amplitude. Even for relatively low intensities (such as at the peak of the first maximum of the probability for scattering from $\hbar k$ to $-\hbar k$ [Fig. 1(c)], an increase

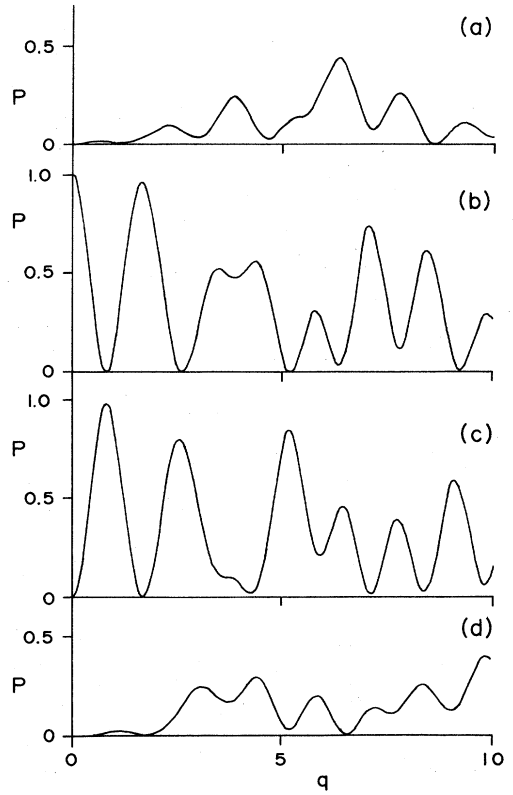

FIG. 2. Scattering probability P as a function of q when $\tau=2$ for scattering from $p_i=\hbar k$ to (a) $p_f=3\hbar k$, (b) $p_f=\hbar k$, (c) $p_f=-\hbar k$, (d) $p_f=-3\hbar k$.

of q by a factor of 2 is sufficient to move the scattering probability to near zero. This sensitivity could explain the difficulty with seeing this effect in the early experiments.[7–12] The variation of the probability with q becomes less severe if β is increased and higher-order scattering is considered. This result is shown in Fig. 3. Although the peak scattering probability is reduced from the case shown in Fig. 2 it is still respectable.

The probability for scattering from $\hbar k$ to $-\hbar k$ and from $3\hbar k$ to $-3\hbar k$ as a function of the interaction time τ is shown in Figs. 4 and 5, respectively, for several values of q. Again, an increase in q causes the probability to fluctuate more rapidly but this time as a function of τ.

The last parameter that can be easily varied in an experiment is the angle between the electron beam and the axis of the standing wave or equivalently β. In Fig. 6 the

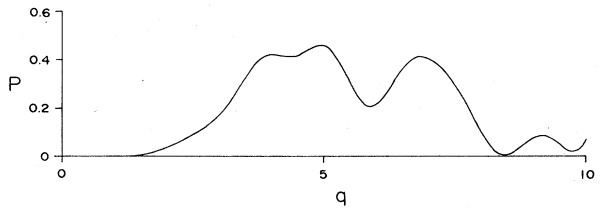

FIG. 3. Scattering probability P as a function of q when $\tau=2$ for scattering from $p_f=-3\hbar k$ to $3\hbar k$.

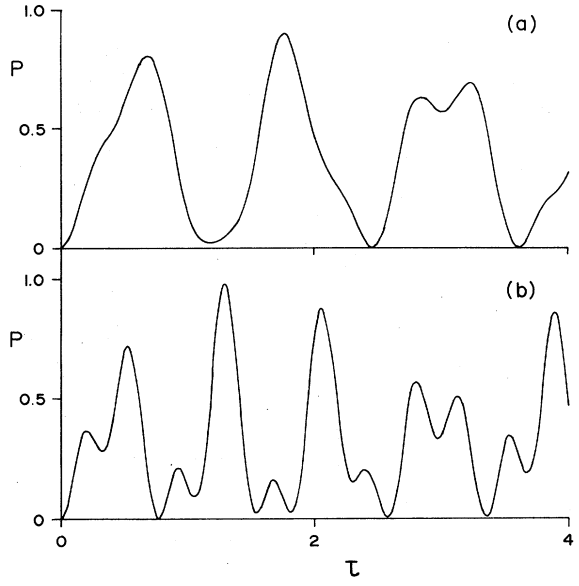

FIG. 4. Scattering probability P as a function of τ for scattering from $p_i=-\hbar k$ to $p_f=\hbar k$ when (a) $q=3$, (b) $q=5$.

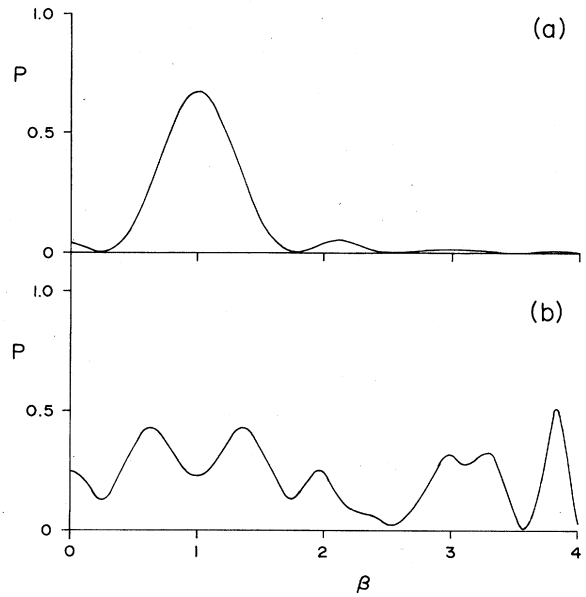

FIG. 6. Scattering probability P as a function of β for the transition from $p_i=\beta\hbar k$ to $p_f=(\beta-2)\hbar k$ when $\tau=2$ and (a) $q=3$, (b) $q=6$.

scattering probability for the transition from $\beta\hbar k$ to $(\beta-2)\hbar k$ as a function of β is shown for two different values of q. These graphs show clearly that for this particular transition the scattering probability is symmetrical about the point $\beta=1$. However, $\beta=1$ is not necessarily the point of maximum scattering. This could also explain the failure of several of the experiments to observe Kapitza-Dirac scattering.

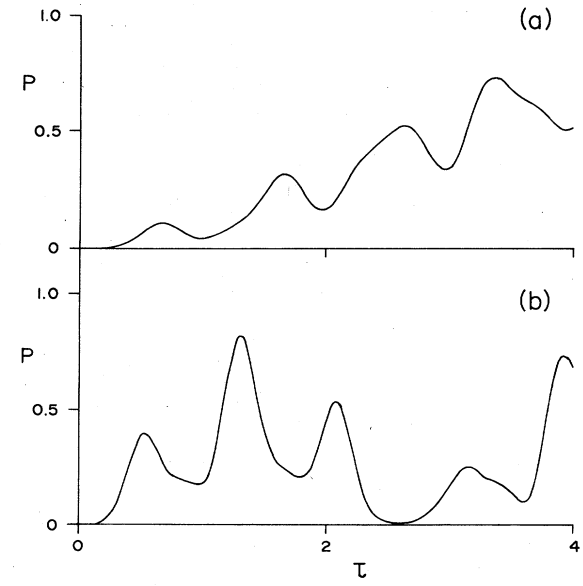

FIG. 5. Scattering probability P as a function of τ for scattering from $p_i=-3\hbar k$ to $p_f=3\hbar k$ when (a) $q=3$, (b) $q=5$.

IV. STRONG COUPLING AND HIGH ENERGY

As was pointed out in Sec. III, the MTS analysis that gave the leading behavior in the form of slowly modulated eigenfunctions of the time-averaged problem breaks down if $q=O(1/\epsilon)$. To examine this case, we employ a variant of the WKB method that allows for the possibility of resonance, caused by the time dependence of the potential. We find that if certain conditions are met between the wave characteristics of the incoming electron wave function and the standing wave field, the amplitude evolves to very large localized maxima near focal points and caustics of the rays of the corresponding classical system. We expect that near such caustics the effects of many-particle interactions and self-radiation will become important. A realistic analysis of these, including relativistic effects, will be presented in a subsequent paper.

This section is organized as follows. In Sec. IV A we discuss the expansion used and outline the calculation. In Sec. IV B we present a perturbative treatment of the Hamilton-Jacobi equation for the rays; a lowest-order resonant case is investigated in detail, and found to lead to the focusing of rays and caustic formation. Finally, in Sec. IV C we analyze the effects of focusing in the classical problem on the probability amplitude.

A. The quasiclassical expansion

Although problems where a high-frequency approximation is relevant have been studied extensively for time-independent potentials very few results of this type exist for the nonseparable time-dependent case.[16] In our discussion of the Kapitza-Dirac problem we shall use the potential

$$V(x,t)=\cos^2 x\cos^2 t$$

but the method can be used for more general time-periodic potentials. For problems of this type we expect quasiperiodic behavior.

In the previous section we saw this quasiperiodicity arise in our MTS treatment which was valid (at least formally) for moderate energies. In examining the high-energy—high-frequency behavior in a system without internal degrees of freedom, whose classical counterpart is described by a Hamiltonian $H(p,q,t)$ the ansatz

$$\psi=A\exp(iS/\hbar) \tag{4.1}$$

is used. Substitution in the Schrödinger equation

$$H(p,q,t)\psi(q,t)=-i\hbar\frac{\partial\psi}{\partial t} \tag{4.2}$$

[where $p\equiv-(i\hbar)(\partial/\partial q)$] results, to leading order in $\hbar^{-1}$ in the equation

$$H\left(\frac{\partial S}{\partial q},q,t\right)+\frac{\partial S}{\partial t}=0\ , \tag{4.3}$$

i.e., the classical Hamilton-Jacobi equation, showing that the phase S corresponds to the classical action. The next order produces an equation for the amplitude ($\Pi=A^2$):

$$\frac{\partial\Pi}{\partial t}+\sum_q\frac{\partial}{\partial q}(v\Pi)=0 \tag{4.4}$$

where

$$v=\frac{\partial H}{\partial p}\left(\frac{\partial S}{\partial q},q,t\right)$$

whose characteristics (rays) are the same as those of (4.3). From (4.4) it is seen that Π is conserved in ray tubes so that if a tube collapses Π (and A) becomes infinite. This happens on caustics of the Hamilton-Jacobi equation which are envelopes of families of rays in the (q,t) plane. If we think, in (q,t,S_q) space, of the surface formed by the rays through some initial curve [e.g., if $S_q(q,0)$ is given], then the caustics are the singularities of its projection on the (q,t) plane, corresponding to folds, etc., of the surface.

It must be understood, as was shown by Buchal and Keller,[17] that the higher-order terms neglected in (4.4) will become large at the caustic and thus they must be included there. This effectively gives rise to a boundary layer in the vicinity of the caustic in which the amplitude is large but still finite. The theory of geometrical optics allows us to connect through a caustic by including an appropriate phase shift in the (complex) amplitude. The value of this phase shift is $m(\pi/2)$ where m is the order of degeneracy of the projection, or equivalently, the number of dimensions lost by the ray tube at the caustic or the change in the number of branches in the vicinity of the point considered. The sign is chosen according to whether the caustic is traversed in the direction of increasing $(-)$ or decreasing $(+)$ S.[18] This was first realized by Keller in his 1958 paper[18] where he also pointed out the need for many-branch descriptions of the form

$$\psi=\sum_{k=1}^{r}A_k\exp(iS_k/\hbar)\ , \tag{4.5}$$

where r is the number of rays through the point under consideration. Demanding single valuedness for ψ he showed that in nonseparable systems, quantum numbers for bound states are in general quarter integers (asymptotically). Maslov[19] proved the asymptotic character of this approximation as $\hbar\to0$ for a special class of problems: essentially "nice" time-independent potentials in several space dimensions and initial conditions that vanish outside some finite region. He showed that for Eq. (4.2) with H in the form

$$H(p,q)=\tfrac{1}{2}\textstyle\sum p^2+V(q) \tag{4.6}$$

and with initial conditions of the form

$$\psi(q,0)=a(q)\exp[ib(q)/\hbar] \tag{4.7}$$

with $a(q)$ zero outside a finite region that an asymptotic expression for ψ is given by

$$\psi(q,t)=\sum_{k=1}^{r}a(q_k)J(q;q_{0k})^{-1/2}\exp\left[\frac{i}{\hbar}S_k(q,t)-i\frac{\pi}{2}\mu_k\right]+O(\hbar)\ . \tag{4.8}$$

Here J is the Jacobian of the mapping from the initial point q_0 of the kth ray through q to the point q induced by the classical Hamiltonian flow, S is the classical action along that ray, and μ_k is the Morse index of the ray, equal to the sum of the numbers m for every encounter of the given ray with caustics. This expression fails at caustics, where the Jacobian becomes singular.

In IV B, we derive a formula similar to (4.8) for our problem, without attempting a rigorous justification (for which we would have to replace our plane-wave initial conditions with a function equal to it inside some finite region, that is large compared to the wavelength, and zero outside). The classical problem for the Kapitza-Dirac potential corresponds to a pendulum with oscillating strength, described by the Hamilton-Jacobi equation

$$S_t+S_x^2+\epsilon^4 q\cos^2 x\cos^2 t=0 \tag{4.9}$$

with initial condition

$$S(x,0)=\epsilon^2\beta x\ . \tag{4.10}$$

Here β is the wavenumber of the incoming plane wave. We consider especially $\beta\simeq O(1/\epsilon^2)$, $q\simeq O(1/\epsilon^2)$. It can be seen that when β is in a rational relation with the periods of the potential the classical problem exhibits resonance (at least, in the sense of perturbation theory), resulting in strong bending of the rays. We treat the case $\beta=1/2\epsilon^2$ (lowest-order resonance) which leads to caustic formation in a time scale of order $1/\epsilon$.

The leading behavior of the rays is found as

$$\begin{aligned}x&=t+C_0(\tau,\xi)+O(\epsilon)\ ,\\ P&=\tfrac{1}{2}+\epsilon(\tfrac{1}{2}C_{0\tau})+O(\epsilon^2)\ ,\end{aligned} \tag{4.11}$$

where C_0 is a solution of the pendulum equation

$$2C_{0\tau\tau}+\sin(2C_0)=0\ ,$$
$$C_0(0,\xi)=\xi, C_{0\tau}(0,\xi)=0\ , \tag{4.12}$$

and ξ, fixed along a ray, gives its position at $t=0$ ($\tau=\epsilon t$ is a slow time scale). We see that the rays are split into groups performing an oscillation around centers which drift with uniform velocity. It is clear from (4.11) and (4.12) that caustics (locations of crossing rays characterized by the condition $x=0$) are cusplike. They form at the center of each group (Fig. 7) and move out, toward the saddle points. New cusps form periodically (with period $T=2\pi/\epsilon$). Every cusp marks an S-shaped folding of the surface $P=P(x,t)$ (Fig. 8) (caustics are singularities of the projection of this surface on the x-t plane, i.e., they correspond to p_x becoming singular). The amplitude is found to leading order as

$$A_0(\tau,\xi)=(C_{0\xi})^{-1/2}+O(\epsilon)\ . \tag{4.13}$$

On caustics $x_\xi=0$ implies $C_{0\xi}=0$ and A appears to become singular. The reason is easy to see if we rewrite the amplitude equation

$$A_t+2PA_x+P_xA-i\epsilon^2A_{xx}=0\ ,$$
$$A(x,0)=1 \tag{4.14}$$

in characteristic coordinates (t,ξ) in which it becomes

$$A_t+\frac{P_\xi}{x_\xi}A-\epsilon^2 i\left[\frac{1}{x_\xi}\frac{\partial}{\partial\xi}\right]^2A=0\ . \tag{4.15}$$

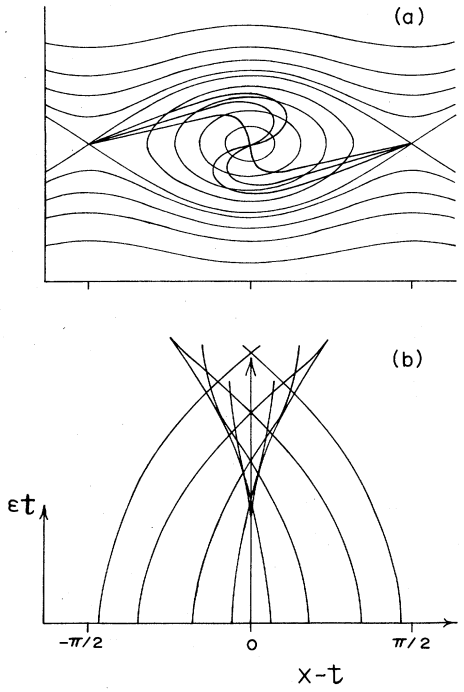

FIG. 7. (a) Phase plane for Eq. (4.12). (b) Rays [Eq. (4.11a)] in x-t plane.

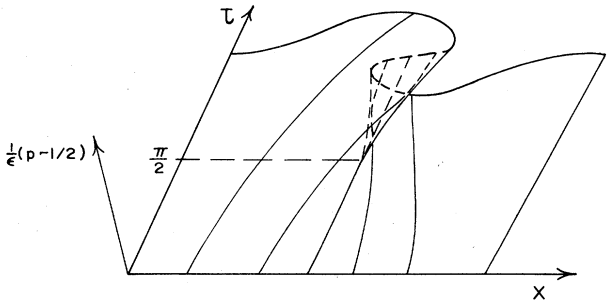

FIG. 8. Surface $P=P(x,t)$ (scales as shown).

Clearly, the $O(\epsilon^2)$ term which was neglected in deriving (4.8) becomes important near the caustic. This suggests studying Eq. (4.15) by techniques of singular perturbation theory. This approach was taken for the reduced wave equation by Buchal and Keller.[17] It produces the correct leading behavior near a smooth convex caustic.

An expansion for ψ is utilized in the form

$$\psi\sim\sum_k A_k\exp(iS_k/\epsilon^2) \tag{4.16}$$

with k ranging over the number of distinct branches of the surface $P=P(x,t)$. Branches join at caustics. A local coordinate system is introduced at caustics and Eq. (4.15) is scaled to give the leading behavior in their vicinity.

Away from cusps, two branches join at a caustic, an incoming and an outgoing branch. Using a matching procedure, we can determine A on the outgoing branch from the (presumed known) values of A on the incoming branch. To carry out the matching it is necessary to analytically continue A into the (classically forbidden) side of the caustic inaccessible to the rays and demand that the amplitude decreases exponentially away from the caustic in this region (Fig. 9). The main result of this analysis, applied to our problem, is that near the caustic the amplitude is of order

$$A\sim O(\epsilon^{-1/6}) \tag{4.17}$$

which might play a very significant role in situations where there exists the possibility of nonlinear interaction (e.g., refractive media, radiation, etc.). The singular perturbation analysis of Eq. (4.15) can be carried out for other cases, e.g., near the "cusp" [which only appears as a cusp in the x-t plane, being a smooth curve on the surface $P=P(x,t)$ itself] or even for higher-dimensional caustics. One can use this approach to get leading-order estimates for the amplitude. To carry out the calculation, especially in higher-dimensional problems, one can be guided into the proper scaling by considering the normal forms that exist giving generic coordinate systems of the surface near various types of caustics (for a classification and a list of normal forms see, e.g., Arnold.[20] This way we can find, for example, that near the cusp

$$A\sim O(\epsilon^{-1/4})\ . \tag{4.18}$$

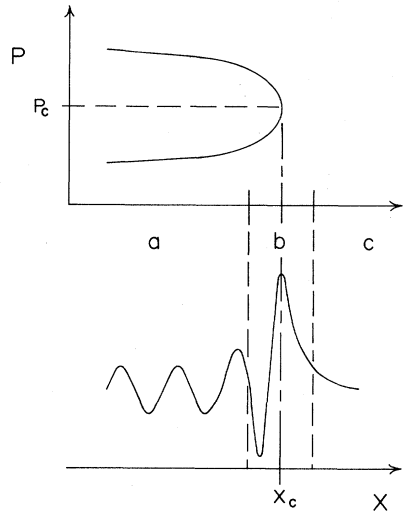

FIG. 9. Structure of solution near a fold of the surface $P=P(x,t)$. (a) Outer (classically accessible) region: solution can be given as the superposition of two waves. (b) Boundary layer: solution is an Airy function oscillating with large amplitude to the left of x_c (caustic) and decaying to the right. (c) Outer (classically inaccessible) region: solution decays exponentially. This is only a schematic, true A complex.

B. The expansion away from caustics

In Sec. III we were led to study a problem of the type

$$\psi_{xx}-qV(x,t'/\epsilon^2)=-i\psi_{t'}\ ,\qquad \psi(x,0)=\exp(i\beta x)\ , \tag{4.19}$$

where

$$\epsilon=\hbar\omega/2mc^2$$

and

$$q=\frac{1}{2}\frac{e^2}{\hbar\omega}A^2\ .$$

Rewriting in terms of $t=t'/\epsilon^2$, Eq. (4.19) becomes

$$\epsilon^2[\psi_{xx}-qV(x,t)\psi]=-i\psi_t\ ,\qquad \psi(x,0)=\exp(i\beta x)\ . \tag{4.20}$$

To keep the discussion general, we leave q as a free parameter. We assume that β has the form

$$\beta=\frac{1}{\epsilon^2}(\beta_0+\sigma\beta_1+\sigma^2\beta_2+\cdots)\ , \tag{4.21}$$

where

$$\sigma=\epsilon^2 q^{1/2}\ .$$

In Sec. III, the case $q=0(1),\beta=0(1)$ was studied. Here we extend the discussion to other interesting limits. Using the ansatz

$$\psi\sim\sum_k A_k\exp(iS_k/\epsilon^2) \tag{4.22}$$

we find that each branch must satisfy (for simplicity we drop subscripts)

$$-\frac{A}{\epsilon^2}(S_t^2+S_x+\epsilon^4 qV)-i(A_t+2S_xA_x+S_{xx}A)+\epsilon^2A_{xx}=0\ . \tag{4.23}$$

We recognize the expression in the first parentheses as the Hamilton-Jacobi equation for the classical action and we resolve the apparent ambiguity in A,S by requiring that S is precisely the classical action, so that

$$S_t+S_x^2+\epsilon^4 qV(x,t)=0\ . \tag{4.24}$$

Then the amplitude A satisfies

$$A_t+2S_xA_x+S_{xx}A-i\epsilon^2A_{xx}=0 \tag{4.25}$$

with initial conditions

$$S(x,0)=\epsilon^2\beta x,\ A(x,0)=1\ . \tag{4.26}$$

It is easily seen that the subcharacteristics of Eq. (4.25) found by neglecting the term $-i\epsilon^2A_{xx}$ are the same as the characteristics (rays) of Eq. (4.24). Since the phase equation is independent of the amplitude it is studied first, to determine the ray structure of the problem.

Utilizing t as the parameter along the rays and ξ as the ray variable, i.e., assuming that the equation of a ray is $x=x(t,\xi)$ with $x(0,\xi)=\xi$, we rewrite Eq. (4.24) in characteristic form,

$$\frac{\partial x}{\partial t}=2P\ , \tag{4.27a}$$

$$\frac{\partial P}{\partial t}=-\sigma^2V_x\ , \tag{4.27b}$$

$$\frac{\partial S}{\partial t}=2P^2+Q=P^2-\sigma^2V\ , \tag{4.27c}$$

$$\frac{\partial Q}{\partial t}=-\sigma^2V_t\ , \tag{4.27d}$$

where we let $S_x=P$, $S_t=Q$, and

$$\frac{\partial}{\partial t}\equiv\frac{\partial}{\partial t}\bigg|_\xi=\frac{\partial}{\partial t}\bigg|_x+\frac{\partial x}{\partial t}\bigg|_\xi\frac{\partial}{\partial x}\bigg|_t$$

is the directional derivative along the rays. The x-P system [Eqs. (4.27a) and (4.27b)] can be solved first, then Eqs. (4.27c) and (4.27d) to allow us to find S.

We shall study Eqs. (4.27a) and (4.27b) using MTS perturbation theory. We expect the weak $[O(\sigma^2)]$ forcing to result in a slow modulation of the rays. The time scale for the lowest-order resonant case we shall consider is

$$\tau=\sigma t,\quad \sigma=\epsilon^2q^{1/2} \tag{4.28}$$

while

$$x=x_0(t,\tau,\xi)+\sigma x_1+\sigma^2x_2+O(\sigma^3)\ ,\qquad P=P_0+\sigma P_1+\sigma^2P_2+O(\sigma^3)\ , \tag{4.29}$$

with initial conditions

$$x(0,\xi)=\xi\ ,\qquad P(0,\xi)=\epsilon^2\beta\simeq\beta_0+\sigma\beta_0+\sigma^2\beta_2+\cdots\ . \tag{4.30}$$

Substituting the expansions (4.29) into (4.27) and equating coefficients of like powers of ϵ we find [recall that $\partial/\partial t=\partial/\partial t+\sigma(\partial/\partial\tau)$] the hierarchy of equations

$$O(1):\ x_{0t}=2P_0,\ P_{0t}=0\ , \tag{4.31}$$

$$O(\sigma):\ x_{1t}=2P_1-x_{0\tau},\ P_{1t}=-P_{0\tau}\ , \tag{4.32}$$

$$O(\sigma^2):\ x_{2t}=2P_2-x_{0\tau},\ P_{2t}=-P_{1\tau}-V_x(x_0,t)\ , \tag{4.33}$$

with initial conditions

$$x_0(0,\xi)=\xi,\ x_i(0,\xi)=0,\ i=1,2,\ldots$$
$$P_i(0,\xi)=\beta_i,\ i=0,1,2,\ldots\ . \tag{4.34}$$

Solving we find

$$O(1):\ x_0=2\beta_0 t+C_0(\tau,\xi),\ C_0(0,\xi)=\xi$$
$$P_0=B_0(\tau,\xi),\ B_0(0,\xi)=\beta_0 \tag{4.35}$$

$$O(\sigma):\ x_1=B_{0\tau}t^2+(2B_1-C_{0\tau})t+C_1(\tau,\xi)\ ,$$
$$P_1=-B_{0\tau}t+B_1(\tau,\xi)\ . \tag{4.36}$$

To ensure that the expansions (4.30) are well ordered (asymptotic) we must disallow unbounded growth in $x_i,P_i,i>0$, so the secular terms must be suppressed. This means $B_{0\tau}=0$, i.e., $B_0=\beta_0$ and

$$B_1=\tfrac{1}{2}C_{0\tau}\rightarrow C_{0\tau}(0,\xi)=2B_1(0,\xi)=2\beta_1\ . \tag{4.37}$$

So far, $C_0(t,\xi)$ has been undermined. To find it we need to consider the next order, $O(\sigma^2)$. We get

$$P_2=\left[-\tfrac{1}{2}C_{0\tau\tau}-\frac{1}{t}\int_0^t V_x(2\beta_0 t+C_0,t)dt\right]t+B_2(\tau,\xi)\ , \tag{4.38}$$

$$B_2(0,\xi)=\beta_2\ .$$

Again, to suppress unbounded behavior in P_2, we demand

$$\tfrac{1}{2}C_{0\tau\tau}+\lim_{t\rightarrow\infty}\int_0 V_x(2\beta_0 t+C_0,t)dt=0 \tag{4.39}$$

(assuming the limit exists, which it does for V periodic in t). The construction can be continued to higher order, but here we only need the leading behavior. We found

$$x=2\beta_0 t+C_0(\tau,\xi)+O(\sigma)\ ,$$
$$P=\beta_0+\sigma(\tfrac{1}{2}C_{0\tau})+O(\sigma^2)\ , \tag{4.40}$$

where C_0 satisfies Eq. (4.39). For the potential

$$V(x,t)=\cos^2 x\cos^2 t\rightarrow V_x=-\cos^2 t\sin(2x) \tag{4.41}$$

we have

$$\int_0^t V_x\,dt=-\frac{1}{8\beta_0}\cos[2(2\beta_0 t+C_0)]$$
$$-\frac{1}{8(2\beta_0\pm1)}\cos\{2[(2\beta\pm1)t+C_0]\}\Big|_0^t$$

a bounded function if $\beta_0\neq0,\pm\frac{1}{2}$. In this case, Eq. (4.39) gives $C_0(\xi,t)=\beta_1\tau+\xi$ and the slow time scale τ only appears as a modification of the fast scale t. In general for a problem of this kind, we should consider a modified fast time scale

$$t^*=\omega(\sigma)t=(\omega_0+\sigma\omega_1+\cdots)t$$

and a slow time scale

$$\tau=\sigma^\alpha t\ ,$$

where α is determined by the type of resonance expected. To treat the simplest case, we consider

$$\beta_0=\tfrac{1}{2},\ \beta_i=0,\ i=1,2,\ldots$$
$$q=\epsilon^{-2},\ \sigma=\epsilon\ . \tag{4.42}$$

Now we have a lowest-order resonance; the integral in Eq. (4.39) becomes

$$\int_0^t V_x dt=\tfrac{1}{4}\cos(2t+2C_0)+\tfrac{1}{16}\cos(4t+2C_0)$$
$$-\tfrac{5}{16}\cos(2C_0)-\frac{t}{4}\sin(2C_0) \tag{4.43}$$

and the equation reads

$$2C_{0\tau\tau}+\sin(2C_0)=0 \tag{4.44}$$

with initial conditions

$$C_0(0,\xi)=\xi,\ C_{0\tau}(0,\xi)=0\ . \tag{4.45}$$

Finally, utilizing Eq. (4.40) in Eq. (4.27c) we find

$$S=\tfrac{1}{4}t+\tfrac{1}{2}C_0+O(\epsilon)\ .$$

Turning our attention to the amplitude equation

$$A_t+2PA_x+P_xA-\epsilon^2 iA_{xx}=0\ , \tag{4.46}$$

we transform it to characteristic coordinates

$$\frac{\partial}{\partial x}=x_\xi^{-1}\frac{\partial}{\partial\xi},\ \frac{\partial}{\partial t}+2P\frac{\partial}{\partial x}=\frac{\partial}{\partial\xi} \tag{4.47}$$

to find

$$A_t+x_\xi^{-1}P_\xi A-\epsilon^2 i\left[x_\xi^{-1}\frac{\partial}{\partial\xi}\right]^2 A=0\ . \tag{4.48}$$

Assuming

$$A=A_0+\epsilon A_1+O(\epsilon^2)$$

and using

$$x_\xi=C_{0\xi}+O(\epsilon)\ ,$$
$$P_\xi=\epsilon(\tfrac{1}{2}C_{0\tau\xi})+O(\epsilon^2)\ ,$$

we have

$$A_t+[C_{0\xi}+O(\epsilon)]^{-1}[\epsilon(\tfrac{1}{2}C_{0\tau\xi})+O(\epsilon^2)]A=O(\epsilon^2) \tag{4.49}$$

from which we find

$$O(1):\ A_{0t}=0\rightarrow A_0=A_0(\tau,\xi)\ \text{ with } A_0(0,\xi)=1\ ,$$

$$O(\epsilon):\ A_{1t}+A_{0t}+\frac{A_0}{C_{0\xi}}\left[\frac{1}{2}C_{0\tau\xi}\right]$$
$$=0\rightarrow A_1=-\left[A_{0\tau}+\frac{1}{2}A\frac{C_{0\tau\xi}}{C_{0\xi}}\right]t$$
$$+(\text{bounded part})$$

so finally, demanding that A_1 be bounded we find that A_0 is given by

$$A_0(\tau,\xi)=(C_{0\xi})^{-1/2}\ . \tag{4.50}$$

Collecting our results we see that the leading asymptotic behavior of ψ is given as

$$\psi\sim[(C_{0\xi})^{-1/2}+O(\epsilon)]\exp\frac{i}{2\epsilon^2}[\tfrac{1}{2}t+C_0(\tau,\xi)+O(\epsilon)] \tag{4.51}$$

with C_0 satisfying Eqs. (4.44)—(4.55). We see (Fig. 7) that the resonance causes a slow-time bending of the rays so that the surface $P=P(x,t)$ given by

$$x=\tfrac{1}{2}t+C_0(\tau,\xi)+O(\epsilon)\ ,$$
$$P=\tfrac{1}{2}+\epsilon(\tfrac{1}{2}C_{0\tau})+O(\epsilon^2) \tag{4.52}$$

develops folds at points where $X_p=0$, or equivalently, $x_\xi=0\rightarrow C_{0\xi}=0$. Clearly this vanishing leads to a blowup in Eq. (4.50), invalidating our asymptotic solution at such points. What went wrong in our analysis is apparent: Eq. (4.48) was analyzed by a regular perturbation expansion, effectively neglecting the $O(\epsilon^2)$ term in finding A. In the neighborhood of envelopes of rays, where $x_\xi=0$, this term clearly will be important so that A would stay finite, although it will become very large there. For this reason points where $x_\xi=0$ are called caustic points. In the next section we complete the picture that this analysis presents by discussing the behavior of solutions in the neighborhood of caustics.

C. The behavior near caustics

We now investigate the behavior of the amplitude near caustics. Following Ludwig,[21] we recognize expansions of the form

$$\psi\sim\sum_k A_k\exp(iS_k/\epsilon^2) \tag{4.53}$$

as resulting when expanding integrals of the type

$$\int A(x,t,z)\exp[iB(x,t,z)/\epsilon^2]dz$$

by the method of stationary phase. Clearly, assuming that ψ can be represented as a superposition of plane waves, we will have, as $\epsilon\rightarrow 0$, contributions from points where

$$B_z=0,\quad B_{zz}\neq 0\ . \tag{4.54}$$

Such points can be shown to correspond to regions away from caustics, where the expansion (4.8) is valid. Smooth caustics correspond to

$$B_{zz}=0,\quad B_{zzz}\neq 0 \tag{4.55}$$

while, finally cusped caustics are found if

$$B_{zzz}=0,\quad B_{zzzz}\neq 0\ . \tag{4.56}$$

Systems of rays in two dimensions have in general singularities of the "fold" (i.e., smooth caustics) or "tuc" (i.e., cusped caustics) kinds (see Arnold[20]), unless some exceptional situations happen. An easy calculation shows that for our problem the worse thing that can happen is a cusp, so we only need to consider the cases for Eqs. (4.55) and (4.56). We can encompass all cases by the exact change of variables

$$B(x,t,z)=S(x,t)+r_1\xi-r_2\xi^2+\frac{\xi_4}{4} \tag{4.57}$$

[with $r_i=r_i(x,t)$] which is related to the normal form that obtains near a tuc singularity [Eq. (56)]. Then, by following the same procedure as in Ref. 21 for the Schrödinger equation, we can construct uniformly valid expansions which, away from caustics, reduce to Eq. (4.8), near smooth caustics have leading terms involving the Airy function and its first derivative and near the cusp involve a generalized Airy function with two arguments and its two first derivatives. By considering the asymptotic behavior of the Airy functions we can find the order of the amplitude in the vicinity of caustics, and by analytically continuing to the classically inaccessible regions we see that our solutions decay exponentially there. Since here we are not as much interested in the detailed behavior at all regimes but mainly on the order of magnitude of the focusing at caustics we shall follow instead a singular perturbation approach due to Buchal and Keller.[17] We introduce boundary layers at caustics and determine the solution in their vicinity by stretching coordinates and matching with the "outer" solution. We shall use this method to give the expansion near a smooth convex portion of the caustic and then we shall give an argument about the behavior near the cusp. We prefer the singular perturbation approach because it would be applicable in a more interesting setting, namely a weakly nonlinear equation that has singularities related to folding subcharacteristics. Clearly, in such cases, it would be impossible to argue from the point of view of integral representations, while the boundary-layer idea would still be relevant (after suitable modification).

We wish to study the equation

$$A_t+\frac{P_\xi}{x_\xi}A-i\epsilon^2\left[\frac{1}{x_\xi}\frac{\partial}{\partial_\xi}\right]^2A=0 \tag{4.58}$$

near characteristics where $x_\xi=0$ and the second-derivative term becomes singular and cannot be ignored. The equation of a ray was found to be (to the leading order)

$$x=t+C_0(\xi,\tau)+O(\epsilon) \tag{4.59}$$

while on caustics $x_\xi=0$, i.e., $C_{0\xi}=0$. We let

$$C_{0\xi}(\xi,\tau)=0\rightarrow\tau=b(\xi)\ ,$$

i.e.,

$$t=\frac{1}{\epsilon}b(\xi)\ . \tag{4.60}$$

Then the equation of a caustic will be

$$x=\frac{1}{\epsilon}b(\xi)+C_0[\xi,b(\xi)]+O(\epsilon)\ . \tag{4.61}$$

With Ref. 17 we introduce a new coordinate system (r,s) where r is arclength along the caustic and s is arclength along the ray tangent at r, measured respectively, from some fixed point and from the point of contact (positive after, negative before) (Fig. 10). An easy calculation shows that, near the caustic [i.e., for $\tau-b(\xi)$ "small"]

$$s\simeq\frac{\sqrt{2}}{\epsilon}[\tau-b(\xi)]+O(1)\ . \tag{4.62}$$

Along the caustic we find (for ξ_0 some fixed number)

$$r\simeq\frac{\sqrt{2}}{\epsilon}[b(\xi)-b(\xi_0)]+O(1) \tag{4.63}$$

while the radius of curvature of the caustic ρ is found from

$$\frac{1}{\rho}\simeq\left.\frac{\epsilon^2(C_{0\xi\tau}+b_\xi C_{0\tau\tau})}{b_\xi[1+(1+\epsilon C_{0\tau})^2]^{3/2}}\right|_{(\xi,b(\xi))}, \tag{4.64}$$

where $C_{0\xi\tau}<0$, $bC_{0\xi\tau}>0$ on the caustic (Fig. 11). The analysis we present here will be valid unless $\rho=\infty$, which happens at the inflection point. The difficulty is not a real one though: it can be resolved by using a better coordinate system, e.g., arclength along the caustic and normal distance from the caustic (our present coordinate system would become rather awkward). In the convex part of the caustic, we set

$$\rho=\frac{1}{\epsilon^2}R(\xi),\quad R=O(1)\ . \tag{4.65}$$

Utilizing Eqs. (4.62) and (4.63), we find that the equation becomes (to leading order)

$$\sqrt{2}A_s+\frac{1}{\sqrt{2}s}A-\frac{i}{\epsilon^2}\frac{R^2}{s}\frac{1}{\sqrt{2}}\frac{\partial}{\partial s}\left[\frac{1}{s}A_s\right]=0 \tag{4.66}$$

all the terms of this equation will be of the same order if we introduce a scaled variable

$$\sigma=\epsilon^{2/3}s\ .$$

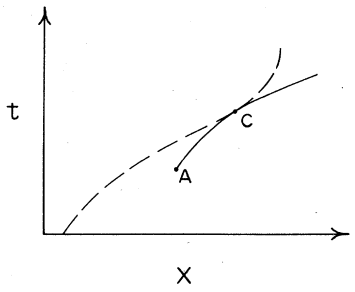

FIG. 10. Ray-caustic coordinates: r is arclength along caustic (solid line), s is arclength along ray (dashed line). At C, $s=0$ while at A, $r=0$.

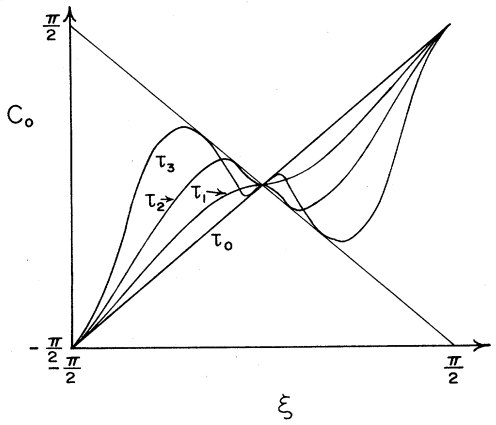

FIG. 11. Snapshots of $C_0(\xi,t)$ as a function of ξ for $0=\tau_0<\tau_1<\tau_2<\tau_3$. At the caustic ($C_{0\xi}=0$) we have $C_{0\xi\tau}<0$, $\operatorname{sgn}C_{0\tau}=-\operatorname{sgn}\xi\ (\mathrm{mod}\,\pi)$ while at $\xi\ (\mathrm{mod}\,\pi)=0$ we have an inflection point ($C_{0\xi\xi}=0$).

Clearly [see Eq. (4.62)], σ is another slow time scale, $\sigma\sim\epsilon^{2/3}[t-b(\xi)/\epsilon]$. In terms of it we have

$$2A_\sigma+\frac{1}{\sigma}A-i\frac{R^2(r)}{\sigma}\frac{\partial}{\partial\sigma}\left[\frac{1}{\sigma}A_\sigma\right]=0\ . \tag{4.67}$$

This equation is analyzed in Ref. 17 where a matching with the outer expansion is carried out to leading order. Adapting the results to our case, we have that the outer solution is

$$A_0=|C_{0\xi}|^{-1/2}\approx|[\tau-b(\xi)]C_{0\xi\tau}(\xi,b)|^{-1/2},\quad \tau<b(\xi)$$

$$A_0=\exp(-\pi i/2)\,|C_{0\xi}|^{-1/2},\quad \tau>b(\xi)$$

[in agreement with Eq. (4.8) and the discussion in Sec. II A], while the inner solution is

$$\tilde{A}\simeq\epsilon^{-1/6}2^{5/6}\pi^{1/2}R^{-1/3}|C_{0\xi\tau}|^{-1/2}\exp\left[\pm\frac{\pi i}{12}\right]\times\mathrm{Ai}\left[-\left[\frac{R}{2}\right]^{2/3}\sigma^2R^{-2}\exp\left[\pm\frac{2\pi i}{3}\right]\right]\exp\left[-i\frac{\sigma^3}{3R^2}\right]$$

(according to $\tau-b\lesseqgtr0$). Therefore we can say that the caustic boundary layer is of thickness $\epsilon^{2/3}$ (in τ) and the amplitude there is

$$A=O(\epsilon^{-1/6})\ .$$

We are able to carry the matching out between incoming and outgoing branches by superimposing

$$\psi=\psi_{\text{incoming}}+\psi_{\text{outgoing}}$$

and demanding that the continuation in the classically inaccessible region is exponentially decaying away from the caustic.

It must be noted that we did not give a uniformly valid expansion but only leading terms near and away from the caustic. As shown in Ref. 21, a uniformly valid expansion contains a term that in the caustic is of $O(\epsilon^{1/6})$ involving the derivative of the Airy function, which in some

intermediate region becomes important and must be included. A similar term should be obtainable by the matched expansion method if we carry it to the next order.

We close this discussion by giving an argument for the order of A at the cusp. By introducing

$$T=\tau-\tau_0,\quad Z=\xi-\xi_0$$

where (τ_0,ξ_0) is the position of the cusp with

$$x_0=\frac{\tau_0}{\epsilon}+C_0(\xi_0,\tau_0)\ ,$$

we have that in the neighborhood of the cusp rays are given by (to leading order)

$$x-x_0\approx\frac{T}{\epsilon}+C_{0\xi\tau}ZT+\tfrac{1}{6}C_{0\xi\xi\xi}Z^3 \tag{4.68}$$

[all functions evaluated at (τ_0,ξ_0)]. Then

$$x_\xi\simeq C_{0\xi\tau}T+\tfrac{1}{2}C_{0\xi\xi\xi}Z^2+\cdots\ ,$$

$$P_\xi\simeq\epsilon\tfrac{1}{2}C_{0\tau\xi}+\cdots\ .$$

It can be easily seen that at (ξ_0,τ_0), $C_{0\xi\tau}<0$, $C_{0\xi\xi\xi}>0$, so that Eq. (4.68) is a surface in (x,Z,T) space that forms a fold at $T=0$ as expected. Transforming the amplitude Eq. (4.58) to (Z,T) we have (letting $C_{0\tau\xi}=a$, $C_{0\xi\xi\xi}=b$)

$$\epsilon A_T+\frac{1}{2}\frac{\epsilon a}{aT+\frac{1}{2}bZ^2}A-i\epsilon^2\left[\frac{1}{aT+\frac{1}{2}bZ^2}\frac{\partial}{\partial Z}\right]^2A=0$$

and all terms are of the same order if we introduce scaled variables

$$T=\epsilon^{2\alpha}\sigma,\quad Z=\epsilon^{\alpha}\zeta\ ,$$

where

$$\alpha=\tfrac{1}{4}\ .$$

We note that $a^\sigma+\frac{1}{2}b\zeta^2=0$ giving the caustic is a smooth curve: its projection in the x-t plane has a cusp at $\sigma=\zeta=0$ $(x=x_0)$. The outer expansion $A_0=(C_{0\xi})^{-1/2}$ becomes, in the vicinity of the cusp

$$A_0\simeq(C_{0\xi\tau}T+\tfrac{1}{2}C_{0\xi\xi\xi}Z^2)^{1/2}$$

and in terms of inner variables

$$A_0\simeq\epsilon^{-1/4}\,|\,a\sigma+\tfrac{1}{2}b\zeta^2\,|^{-1/2}\ . \tag{4.69}$$

Without being proof that $A\simeq O(\epsilon^{-1/4})$ near the cusp, it still provides us with a good indication as to what we can expect there (it turns out to be what we would get if we carried out Ludwig's method for this case).[22]

V. CONCLUSION

In this paper we have investigated Kapitza-Dirac scattering in two experimentally interesting regimes of values of the governing parameters q, β, and ϵ. The choice of two regimes was based on the distinct methods used for their study as well as qualitative differences in the behavior of the system. Previous analyses of this problem led to conflicting answers on the role of the time-dependent portion of the Hamiltonian. In the first case examined in this paper $(q,\beta\ll\epsilon^{-1})$ it was found that, to lowest order, treatment of a time-averaged problem was basically justified. Time-dependent effects were essentially of two types: a small $[O(\epsilon q)]$, rapidly oscillating correction to the amplitude and an $O(\epsilon^2)$ correction to the frequency. This analysis breaks down as q or β becomes comparable to ϵ^{-1}. We showed how to treat a representative case using a semiclassical approximation. It was found that again the behavior is quasiperiodic but under certain "resonance" conditions focusing leading to large localized maxima of the probability amplitude is possible.

One obvious advantage of an analysis in the weak coupling regime is that it provides a solution that can easily be used to study scattering probability as a function of any of the parameters. For instance, we provide the first study of the sensitivity of scattering to the standing-wave intensity. From our studies, we find the inconclusive results of the experiments not to be surprising.

Finally, we would like to point out the general applicability of the perturbation technique employed in our study to other nonseparable time-dependent situations. In fact, the qualitative aspects of our results only depend on the potential being a periodic function of space and time. Extension to more general time periodic potentials is straightforward. Of course, results of this type are subject, especially in their long-time applicability, to the limitations of multiple-scaling perturbation theory.

[1] P. L. Kapitza and P. A. M. Dirac, Proc. Cambridge Philos. Soc. **29**, 297 (1933).

[2] L. S. Bartell, J. Appl. Phys. **38**, 1561 (1967).

[3] H. De Lang, Opt. Commun. **4**, 191 (1971).

[4] M. V. Fedorov, Zh. Eksp. Teor. Fiz. **52**, 1434 (1967) [Sov. Phys.—JETP **25**, 952 (1967)].

[5] R. Gush and H. P. Gush, Phys. Rev. D **3**, 1912 (1971).

[6] F. Ehlotzky and C. Leubner, J. Phys. A **8**, 1806 (1975).

[7] H. Schwarz, H. A. Tourtellotte, and V. W. Gaertner, Phys. Lett. **19A**, 202 (1967).

[8] H. Schwarz, Z. Phys. **204**, 276 (1967).

[9] L. S. Bartell, R. R. Roskos, and H. B. Thompson, Phys. Rev. **166**, 1494 (1968).

[10] Y. Takeda and I. Matsui, J. Phys. Soc. Jpn. **25**, 1202 (1968).

[11] H.-Chr. Pfeiffer, Phys. Lett. **26A**, 362 (1968).

[12] H. Schwarz, Phys. Lett. **43A**, 457 (1973).

[13] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics* (Springer, New York, 1981).

[14] J. Meixner and F. W. Schafke, *Mathieusche Funcktionen and Spharoidfuncktionen* (Springer, Berlin, 1954). See also Gertrude Blanch, in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, D.C., 1965), pp. 721—750 (especially p. 744).

[15] For a discussion of the Fredholm Alternative Theorem as it applied to MTS perturbation methods, see P. C. Fife, Bull. Am. Math. Soc. **84**, 693 (1978).

[16] See, for example, F. Gesztesy and H. Mitter, J. Phys. A **14**, L79 (1981); K. Yajima, J. Math. Soc. Jpn. **29**, 729 (1977); D. R. Yafaev, Funct. Anal. Appl. **14**, 325 (1980).

[17]R. N. Buchal and J. B. Keller, Commun. Pure Appl. Math. **8**, 85 (1960).
[18]J. B. Keller, Ann. Phys. (N.Y.) **4**, 180 (1958).
[19]For an overview, see V. P. Maslov and M. Fedoriuk, *Semi-Classical Approximation in Quantum Mechanics* (Reidel, Boston, 1981).
[20]V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978).
[21]D. Ludwig, Commun. Pure Appl. Math. **19**, 215 (1966).
[22]For a modern account of the treatment of caustics using the ideas of singularity theory, see V. Guillemin and S. Sternberg, *Geometric Asymptotics* (American Mathematical Society, Providence, 1977).