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On the Calculus of Variations and Its Major Influences on the Mathematics of the First Half of Our Century. Part II.*

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Note: The first part of this paper (sections 1–5) appeared in 1994, in the August-September issue of the *Monthly* (pp. 674–678). What follows is a short summary of the first part and the concluding part of the paper.

Summary of Sections 1–5. The calculus of variations deals with the problem of determination of extrema of functionals given by integrals. It may be said to have begun with Johann Bernoulli's brachystochrone problem of 1696. The birthyear of its *theory* is 1744, the year in which Euler published his *Methodus inveniendi* . . . that included his necessary condition for a minimum and a splendid collection of problems.

Lagrange went beyond Euler by inventing the "method of variations" and the concept of the first variation δJ of a functional J; δJ is the analogue of the first derivative of a function.

Both Euler and Lagrange contributed to the problem of minimal surfaces, an important geometric application of the calculus of variations.

The discovery of sufficient conditions for an extremum of a functional was due to Jacobi and Weierstrass. Legendre introduced the concept of the second variation $\delta^2 J$ of a functional J, which is an analog of the second derivative of a function. Weierstrass rigorized the calculus of variations and introduced the fundamental concepts of a field of extremals and of the E-function, "a turning point in the history of the calculus of variations."

6. IMPACT ON EARLY FUNCTIONAL ANALYSIS. The proverbial *Weierstrassian rigor* (Felix Klein's term of 1885) had a profound influence on the theories of *functionals* and *function spaces* of our century. In fact, near the end of the last century, the central role of functionals in the calculus of variations may very well have directed attention to functionals in general. This may have been a kind of subliminal influence that affected primarily the younger generation represented by Volterra and later by Fréchet and F. Riesz.

*Abbreviated version of a paper with the same title.

Five notes of 1887 by Volterra on special classes of functionals, investigated as concepts *per se*, marked the *birth of functional analysis*. Influenced by Dini and Betti, the latter a close friend of Riemann's, Volterra wanted to *generalize complex analysis*. His whole theory was based on the calculus of variations.

Hadamard was enthusiastic about Volterra's novel theory and contributed a basic result that attained final form in a celebrated paper by F. Riesz. (Riesz's theorem states that every continuous linear functional $U[f]$, $f \in C[a,b]$, can be expressed as a Stieltjes integral

$$U[f] = \int_a^b f(x)\, dw(x),$$

where $w(x)$ is determined by U and is of bounded variation on $[a,b]$.) Hadamard pointed to the variational-analytic roots of functional analysis—the *calculus* gave rise to the *calculus of variations*, which in turn produced the *functional calculus*, later called functional analysis.

Hadamard regarded the calculus of variations as

> "a first chapter of functional calculus, whose development will without doubt be one of the first tasks in the analysis of the future."

(This quotation is from Hadamard's book on the calculus of variations published in 1910.)

7. DIRICHLET'S PRINCIPLE. Besides its direct impact on developing functional analysis from 1887 to about 1903 the calculus of variations also had an indirect and, in the long run, an even greater impact on (classical and) functional analysis, an impact which dates back to about 1870 and reached functional analysis shortly after the turn of the century. This involved *partial differential equations*, for which the development proceeded from the search for general solutions to solution formulas for boundary and initial value problems (Green, Poisson, Kirchhoff, etc.) and on to existence (and uniqueness) proofs, notably for Laplace's equation for a function u of three variables

$$\triangle u = u_{xx} + u_{yy} + u_{zz} = 0 \tag{7.1}$$

(or in two variables x, y), which is basic in gravitation, electrostatics, stationary heat conduction, and fluid flow. For the corresponding *Dirichlet problem* in a general domain,

$$\triangle u = 0 \text{ in } \Omega; \quad u|_{\partial\Omega} = f, \quad u \in C^2(\Omega) \cap C^0(\overline{\Omega}), \quad \Omega \subset \mathbb{R}^2 \text{ or } \mathbb{R}^3, \tag{7.2}$$

a proof of existence of a solution u is not easy and attracted the efforts of many of the greatest mathematicians for quite some time. A (faulty) method of proof was provided by a principle taken from the *calculus of variations*. B. Riemann (1826–66) had first seen it in lectures of G. Lejeune Dirichlet (1805–59) in Berlin and named it after him:

Dirichlet's principle. There exists a function u that minimizes the functional (the so-called *Dirichlet integral*)

$$D[u] = \int_\Omega |\text{grad}\, u|^2\, dV, \qquad \Omega \subset \mathbb{R}^2 \text{ or } \mathbb{R}^3, \tag{7.3}$$

among all functions $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ which take on given values f on the boundary $\partial\Omega$ of Ω, and that function u satisfies (7.2).

Note that Laplace's equation is the Euler equation for (7.3).

We see that the Dirichlet integral is bounded below (by zero) and the claim of existence of a minimum is based on a conceptual error, namely the failure to distinguish *greatest lower bound* from *minimum*.

Dirichlet's principle was used earlier (in 1839) by C. F. Gauss (1777–1855) in his potential-theoretic investigations. Gauss claimed that if V is the potential of a mass distribution of density m on the surface S of a spatial region and U is a given function on S, then among all possible distributions there *obviously* exists one for which

$$\int (V - 2U) m \, dS$$

takes its minimum. If this is granted, then one can show, as Gauss did, that: (i) for this minimizing distribution, $V - U = const$ at all points of S that carry mass; (ii) if $U = 0$, there must be mass everywhere on S; (iii) one can obtain a distribution whose potential V on S equals U. Thus one could conclude the existence of the required *harmonic function* [a twice *continuously* differentiable solution of (7.1)] as well as its representation as the potential of a single layer of mass.

After Gauss, Dirichlet's principle was used in 1847 by W. Thomson (Lord Kelvin, 1824–1907) in order to "prove" the existence of a solution u of the differential equation

$$\left(\alpha^2 u_x\right)_x + \left(\alpha^2 u_y\right)_y + \left(\alpha^2 u_z\right)_z = 4\pi\zeta \tag{7.4}$$

that vanishes at infinity. Here α and ζ are given functions and ζ is zero outside a given bounded region. Because of this work, the principle is usually called in England *Thomson's principle*.

After attending Dirichlet's lectures in Berlin, Riemann used the principle in his famous thesis of 1851 as a key tool for obtaining fundamental results on *complex* analytic functions from *real* potential theory. Since the principle does not always hold, some of Riemann's proofs were not complete. All the results, however, turned out to be correct and were proved later by other methods. The same holds for Riemann's later use of Dirichlet's principle in his monumental paper on the theory of Abelian functions.

It was a strange situation. Dirichlet's principle had helped to produce exciting basic results but doubts about its validity began to appear, first in private remarks of Weierstrass—which did not impress Riemann, who placed no decisive value on the derivation of his existence theorems by Dirichlet's principle—and then, after both Dirichlet and Riemann had died, in Weierstrass's public address to the Berlin Academy:

> "From Dirichlet's assumptions it can only be claimed that for (7.3) there exists a certain lower bound to which (7.3) can come arbitrarily close, without being forced to actually reach it. Dirichlet's argument appears invalid."

8. ANOTHER IMPACT ON FUNCTIONAL ANALYSIS. The breakdown of Dirichlet's principle had an enormous positive effect on analysis because it led to the creation of three ingenious new methods for obtaining existence proofs for the Dirichlet problem—by H. A. Schwarz (1843–1925), H. Poincaré (1854–1912), and C. Neumann (1832–1925)—as well as to the development of the direct methods of the calculus of variations initiated by D. Hilbert (1862–1943).

C. Neumann's *method of the arithmetic mean* (1870) was of great importance to analysis and to functional analysis because it sparked work on spectral theory, integral equations, and through it on Hilbert spaces. Neumann assumed the solution to be the potential of a *double layer*, a layer of dipoles normal to the boundary surface (or curve) $\partial\Omega$, of unknown density $\sigma(Q)$, $Q \in \partial\Omega$. Writing this potential in the form

$$u(P) = \frac{1}{2\pi}\int_{\partial\Omega}\sigma(Q)\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right) dS(Q), \quad r = d(P;Q), \quad Q \in \partial\Omega, \tag{8.1}$$

(ν the outer normal of $\partial\Omega$) one obtains for the unknown density σ the integral equation

$$\sigma(Q) - \frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial}{\partial\nu}\left(\frac{1}{r^*}\right)\sigma(Q^*)\, dS(Q^*) = \varphi(Q),$$

$$\varphi(Q) \text{ the given values of } u \text{ on } \partial\Omega,\ r^* = d(Q,Q^*). \tag{8.2}$$

Because of the term $\sigma(Q)$, the operator form of this "integral equation of the second kind" is

$$(I + K)\sigma = \varphi. \tag{8.3}$$

Its solution should obviously be

$$\sigma = (I + K)^{-1}\varphi = \varphi - K\varphi + K^2\varphi - K^3\varphi + \cdots . \tag{8.4}$$

Using this idea, Neumann was able to prove existence of a solution of the Dirichlet problem by integral equation methods. He solved (8.2) by successive approximation, defining $\sigma_0 = \varphi$ and

$$\sigma_n = (-K)\sigma_{n-1} = -\frac{1}{2\pi}\int_{\partial\Omega}\sigma_{n-1}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right) dS = (-K)^n\varphi.$$

This gave him the *Neumann series* (8.4) as the solution of the problem in a convex domain in space or in the plane.

We conclude this part of the section with a short list of events resulting from Neumann's work. In 1888, Weierstrass's former student du Bois-Reymond coined the term *integral equations* and expressed the desirability of a general theory of these equations, with which one could solve various problems such as that solved by Neumann. It did not take long for such theories—by Le Roux (1895), Volterra (1896), and the most famous one by Fredholm (1900, 1903)—to appear. Hilbert "caught fire at once" (as H. Weyl put it) and developed his spectral theory of integral equations with symmetric kernel published in six *Mitteilungen* between 1904 and 1910. Most important of these was the fourth *Mitteilung*, the earliest truly functional-analytic treatment of integral equations, in which he introduced continuous and compact (Hilbert said "completely continuous") forms (cast into operator language by F. Riesz in 1913).

After the breakdown of Dirichlet's principle there was no more *general* principle for handling various problems of applied mathematics. It seems that in that situation Hilbert first put his hope in the calculus of variations, which had produced general principles in the past. In Problem 23 of his famous talk of 1900 on unsolved problems Hilbert had drawn attention to Weierstrass's work and to A. Kneser's book, the first presentation of the modern calculus of variations. Not intimidated by Weierstrass, he was able to re-establish the Dirichlet principle within proper limits as a valid method of proof. He did this in two papers of 1900 and 1901 (reprinted 1905). In the first of these papers he proposed the following more general formulation of Dirichlet's principle.

> "Every regular problem of the calculus of variations [Sec. 3] has a solution as soon as suitable restrictive assumptions with respect to the nature of the given boundary conditions are satisfied and, if necessary, the concept of a solution is suitably generalized."

This approach gave rise to the *direct methods* (methods without the use of the Euler-Lagrange equations), which became of basic importance in the existence theory of the calculus of variations. (A forerunner of these methods was Euler's almost forgotten "direct difference method".) Another solution of the Dirichlet problem by direct methods was given later (in 1907) by Lebesgue.

Apart from these splendid initial steps, Hilbert made no further attempts to uniformize analysis by methods of the calculus of variations. Instead, he turned to integral equations, perhaps as a more promising tool for the same purpose. But his idea of weakening the notion of solutions became a guiding principle in the calculus of variations of our century.

9. PLATEAU'S PROBLEM. By *Plateau's problem* one means the determination of a simply connected portion of a minimal surface S in R^3 bounded by a given curve in space. This problem is named after the Belgian physicist J. Plateau, who realized minimal surfaces experimentally by dipping wires (the boundary curves) into soap solutions, minimum area corresponding to minimum surface energy.

Plateau's problem has attracted great attention from around 1870 to the present. It is a problem genuinely belonging to the calculus of variations. The solution methods developed by a pleiad of researchers (Schwarz, Lebesgue, Korn, Bernstein, Haar, Garnier, Radó, Douglas, Courant, McShane) were interrelated with various branches of the mathematics of our century, which they fertilized immensely.

From a more general viewpoint we can regard the evolution of the theory of minimal surfaces and of Plateau's problem as particular cases of the development of the theory of partial differential equations. The first stage concerned special solutions of the minimal surface equation (special minimal surface), while the second stage dealt with general solutions (Weierstrass's general solution formulas). The third stage, the solution of boundary value problems (Plateau's problem), began in 1867 with Schwarz's work, slightly later than work on partial differential equations in general. For this work, Kelvin, Gauss, Riemann, and others had already switched from general solutions to the geometrically and physically more useful boundary and initial value problems.

10. GLOBAL CALCULUS OF VARIATIONS (MORSE THEORY). We have seen that early functional analysis owed much to the calculus of variations, and that *general topology* (*set-theoretic topology*) developed along with it, in a process of mutual give-and-take that extended over the first three decades of our century. This led to the creation of modern nonlinear analysis in connection with partial differential equations, and even more in connection with the calculus of variations, resulting in the so-called *calculus of variations in the large*, or *Morse theory*, for short.

In this calculus one is concerned with relations between properties of a "space" X (usually a topological space, often a Riemannian manifold) and a real-valued continuous function f defined on X.

The beginnings of this fascinating theory are due to Poincaré, who began his work on periodic solutions of the differential equations of celestial mechanics in his thesis of 1879 and knew of "Morse inequalities" for a surface as early as 1885, seven years before Morse was born.

The next stage of the development can perhaps best be seen from G. D. Birkhoff's book on Dynamical Systems (1927). In it Birkhoff emphasized the growing importance of topology in the calculus of variations. He also mentioned his Ph.D. student M. Morse, who worked out his calculations in the large in his book published by the AMS in 1934. "In the large" meant that Morse considered the whole manifold on which the variational problem was given and not just in a small neighborhood of an extremal ("calculus of variations in the small"). In the Preface he commented:

> "Any problem which is nonlinear in character, which involves more than one coordinate system or more than one variable, or whose structure is initially defined in the large, is likely to require consideration of topology..."

Around 1930 topology was developing rapidly and eliciting general interest. This was evidenced, for example, by the wealth of new results contained in Alexandroff-Hopf's *Topologie I* of 1935 and by the papers presented at the Moscow Topology Congress of 1935. Thus it was just the right time for a marriage of the classical calculus of variations and topology, and Morse made ingenious use of the latter.

Define a *critical point* of a smooth function f on a smooth manifold M to be a point p at which

$$\operatorname{grad} f = 0,$$

Morse classified the critical points in terms of the eigenvalues of the Hessian matrix

$$H(f,p) = \left[\frac{\partial^2 f}{\partial x_i \, \partial x_j}\right]_p \tag{10.1}$$

and obtained *topological* lower bounds (in terms of Betti numbers) for the number of critical points. These are the famous *Morse inequalities*. This work applied to *functions*. From *functions*, Morse proceeded to *functionals*, essentially n-dimensional analogs of our integral (2.1) and generalizations. Now for *functions* the topology at that time was sufficient. For *functionals*, that is, functions on a space of curves, Morse had to develop a topology in function spaces. Instead of critical *points* he had *critical curves*. He described this intuitively in his talk of 1932 at the Zurich International Congress. From it, one gains the impression that in Morse's theory the calculus of variations and topology had developed multiple relations. Incidentally, a similar theory was created simultaneously and independently by L. A. Lyusternik and L. Schnirelman. This shows that at certain times certain things are *in the air*, in the sense that problems which extend known settings in a natural way become accessible as soon as basic theories (topology in the present case) have been sufficiently developed.

Let me conclude with the following remark. We started with the calculus, sketched briefly how calculus evolved into the calculus of variations, and outlined the most important ways in which the calculus of variations accompanied, influenced, or even initiated progress in various parts of analysis, geometry, functional analysis and, finally, topology. The whole process was heterogeneous, but I hope that we have seen traces of some intrinsic logic of the development here and there. The presentation can perhaps help to bring about a better understanding of certain features of present-day mathematics. The calculus of variations seems to have had a profound effect on the general development of mathematics. I hope that readers will see this as an invitation to further research pertaining to details as well as to larger issues that call for a more profound study than that presented in these pages.

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> I do hate sums. There is no greater mistake than to call arithmetic an exact science. There are permutations and aberrations discernible to minds entirely noble like mine; subtle variations which ordinary accountants fail to discover; hidden laws of number which it requires a mind like mine to perceive. For instance, if you add a sum from the bottom up, and then from the top down, the result is always different.
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> —*Mrs. La Touche*
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> *Mathematical Gazette*, vol. 12.