# Hints for problem 2.9.6

Let $A(t)$ be a continuous real valued square matrix.

1. Show that every solution of the nonautonomous linear system

   $$\mathbf{x}' = A(t)\mathbf{x}$$

   satisfies

   $$|\mathbf{x}(t)| \leq |\mathbf{x}(0)| exp \int_0^t ||A(s)|| ds \ .$$

   **Hint:** use the integral equation form:

   $$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t A(s)\mathbf{x}(s) ds$$

   to arrive at the inequality

   $$|\mathbf{x}(t)| \leq |\mathbf{x}(0)| + \int_0^t ||A(s)|| |\mathbf{x}(s)| ds$$

   From this it is straightforward to arrive at the desired inequality, using an extension of the Gronwall lemma proved in class:
   **Lemma:** let $f(t)$, $g(t)$ be continuous, nonnegative for $t \geq 0$ and $c > 0$ and assume

   $$f(t) \leq U(t) := c + \int_0^t f(s)g(s) ds \ .$$

   Then

   $$f(t) \leq c e^{\int_0^t g(s) ds} \ .$$

   **Proof:** By continuity of $f$, $g$, it follows that $U$ is differentiable:

   $$U'(t) = f(t)g(t) \leq U(t)g(t) \Rightarrow U' - Ug \leq 0 \Rightarrow \frac{d}{dt}\left(U e^{-\int_0^t g(s) ds}\right) \leq 0$$

   so that

   $$f(t) \leq U(t) \leq U(0) e^{\int_0^t g(s) ds} = c e^{\int_0^t g(s) ds} \ .$$

2. Show that if $\int_0^\infty ||A(s)|| ds < \infty$ then every solution of the system in part (1) has a finite limit as $t$ approaches infinity.
   **Hint:** Show that part (1) implies that $|\mathbf{x}(t)| < M$ where $M$ is some positive constant. Then show that

   $$|\mathbf{x}(t_n) - \mathbf{x}(\hat{t}_n)| \leq \left| \int_{\hat{t}_n}^{t_n} ||A(s)|| |\mathbf{x}(s)| ds \right| \leq M \left| \int_{\hat{t}_n}^{t_n} ||A(s)|| ds \right|$$

so that, if $t_n$ and $\hat{t}_n$ are two sequences approaching infinity, and $T_n = min(t_n, \hat{t}_n)$ then

$$|\mathbf{x}(t_n) - \mathbf{x}(\hat{t}_n)| \leq M \int_{T_n}^{\infty} ||A(s)||ds \to 0$$

as $n \to \infty$ (why?). Use this to conclude that $\mathbf{x}(t)$ must have a unique limit point as $t \to \infty$ over any sequence.