A FINITE REPRESENTATION FOR e^{tA}.

by Walter Thomas Kyner

Let A be an $n \times n$ matrix with scalar entries. The Cayley-Hamilton theorem states that A satisfies its characteristic equation, i.e., if

$$p(\lambda) = det(\lambda I - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n,$$

then

$$p(A) = A^n + p_1 A^{n-1} + \cdots + p_{n-1}A + p_n I = 0,$$

where I denotes the identity matrix. It follows that the matrix

$$e^{tA} = \sum_{m=0}^{\infty} \frac{t^m A^m}{m!}$$

has a finite representation,

$$e^{tA} = u_1(t)A^{n-1} + u_2(t)A^{n-2} + \cdots + u_{n-1}(t)A + u_n(t)I.$$

The purpose of this note is to show that the $\{u_j\}$ are analytic functions of t and to present an elementary procedure for constructing them. In addition, we shall exhibit a second finite representation of e^{tA} that is based on Laplace transform theory..

We first note that $e^{tA} = \Phi(t)$, the unique solution to the matrix differential equation,

$$D\Phi(t) = A\Phi(t) \qquad \text{such that } \Phi(0) = I.$$

In this differential equation, D denotes the derivative with respect to t. It follows that, for any integer k,

$$D^k \Phi(t) = A^k \Phi(t).$$

And since $\Phi(0) = I$,

$$D^k \Phi(0) = A^k.$$

These relations play a key role in the development of a procedure for constructing the coefficient functions, $\{u_j\}$.

POSTPONE

Remark: Let $T(s) = \mathcal{L}(\Phi(t))$ denote the Laplace transform of the solution to the initial value problem,

$$D\Phi = A\Phi \quad , \Phi(0) = I.$$

then is is easy to show that $T(s) = (sI - A)^{-1}$. Furthermore, the entries of $T(s)$ are linear combinations of

$$\frac{s^k}{p(s)} = \mathcal{L}(D^k w(t)) \quad k = 0, 1, 2, \ldots n - 1.$$

This observation led to this study of a finite representation of $\Phi(t) = e^{tA}$.

START AGAIN

The matrix function $\Phi(t)$ also is a solution to a differential equation associated with $p(\lambda)$, the characteristic polynomial of the constant matrix A, namely, if $L \doteq p(D)$, then

$$L\Phi(t) = p(D)\Phi(t) = [D^n + p_1 D^{n-1} + \cdots + p_{n-1} D + p_n I]\Phi(t) = 0.$$

The proof is transparent:

$$L\Phi(t) = p(D)\Phi(t) = p(A)\Phi(t) = 0$$

since $p(A) = 0$.

Therefore, we require that the coefficient functions, $\{u_j(t)\}$, also be solutions of this differential equation, i.e., $Lu_j = 0, \quad j = 1, 2, \ldots n$. This also ensures that they are analytic functions of t. Furthermore, if

$$\Phi(t) = u_1(t)A^{n-1} + u_2(t)A^{n-2} + \cdots + u_{n-1}(t)A + u_n(t)I$$

is a solution to the matrix differential equation, $D\Phi(t) = A\Phi(t)$, then we must have

$$\begin{aligned} D\Phi(t) - A|Phi(t) &= Du_1(t)A^{n-1} + Du_2(t)A^{n-2} + \cdots + Du_{n-1}(t)A + Du_n(t)I \\ &\quad -u_1(t)A^n + u_2(t)A^{n-1} + \cdots + u_{n-1}(t)A^n + u_n(t)Aa = 0. \end{aligned}$$

By virtue of the Cayley-Hamilton theorem, we can set

$$-A^n = p_1 A^{n-1} + \cdots + p_{n-1}A + p_n I$$

and get the following constraint on the coefficient functions,

$$\begin{aligned} 0 &= [Du_1(t) + p_1 u_1(t) - u_2(t)]A^{n-1} + [Du_2(t) + p_2 u_1(t) - u_3(t)]A^{n-2} \\ &\quad + \cdots + [Du_{n-1}(t) + p_{n-1}u_1(t) - u_n(t)]A + [Du_n(t) + p_n u_1(t)]I. \end{aligned}$$

Clearly, this can be satisfied by setting

$$u_{k+1}(t) = Du_k(t) + p_k u_1(t), \quad k = 1, 2, \ldots n-1$$

and requiring that

$$Du_n(t) + p_n u_1(t) = 0.$$

Fortunately, these conditions are the formulas obtained by writing the scalar differential equation, $p(D)u_1(t) = 0$, in nested form (*Horner's algorithm.* For example, if $n = 4$,

$$\begin{aligned} p(D)u_1(t) &= p_4 u_1(t) + p_3 D u_1(t) + p_2 D^2 u_1(t) + p_1 D^3 u_1(t) + D^4 u_1(t) \\ &= p_4 u_1(t) + D[p_3 u_1(t) + D[p_2 u_1(t) + D[p_1 u_1(t) + D u_1(t)]]]. \end{aligned}$$

We have shown that if u_1 is any solution to the scalar differential equation, $p(D)u = 0$, then the other coefficient functions $\{u_j\}$ can be defined so that

$$\Phi(t) = u_1(t)A^{n-1} + u_2(t)A^{n-2} + \cdots + u_{n-1}(t)A + u_n(t)I$$

is a solution to the matrix differential equation, $D\Phi = A\Phi$. We have left the problem of picking the initial values $D^k u_1(0), \quad 0 \le j \le n-1$.

It is a straightforward task to derive the initial conditions for the full set of coefficient functions. From

$$\Phi(t) = u_1(t)A^{n-1} + u_2(t)A^{n-2} + \cdots + u_{n-1}(t)A + u_n(t)I,$$

and $D^k\Phi(t) = A^k\Phi(t)$, we have

$$A^k\Phi(t) = D^k u_1(t)A^{n-1} + D^k u_2(t)A^{n-2} + \cdots + D^k u_{n-1}(t)A + D^k u_n(t)I, \quad k = 0, 1, 2, \ldots n-1.$$

We next set $t = 0$ to get

$$A^k = D^k u_1(0)A^{n-1} + D^k u_2(0)A^{n-2} + \cdots + D^k u_{n-1}(0)A + D^k u_n(0)I, \quad k = 0, 1, 2, \ldots n-1.$$

After matching powers of A, we obtain

$D^k u_j(0) = 0, \ \ , k \neq j-1$, and $D^{j-1} u_j(0) = 1, \ \ k = 0, 1, 2, \ldots n-1. \ \ j = 1, 2, \ldots n.$

Remark: The proof implies that if $t = 0$, then the Wronskian matrix

$$W(u_1, \ldots, u_n) = I.$$

At this point some care is needed. We have defined the coefficient function $u_j, 2 \leq j \leq n$ in terms of u_1, but we also have required that they be solutions to the scalar differential equation $p(D)u = 0$ with the above initial conditions. Hence, we must establish that they are not overdetermined.

We first note that since $u_2(t) = Du_1(t) + p_1 u_1(t)$, it follows that $D^k u_2(t) = D^{k+1} u_1(t) + p_1 D^k u_1(t), \ \ 1 \leq k \leq n-1$. Invoking the initial values of $u_1(t)$, we obtain $u_2(0) = 0, \ \ D^k u_2(0) = 0, \ \ 1 \leq k \leq n-3, \ \ D^{n-2} u_2(0) = 1$, and $D^{n-1} u_2(0) = D^n u_1(0) + p_1 u_1(0) = 0$. Therefore $u_2(t)$ is well defined. A similar argument can be used to justify the definitions of the other coefficient functions.

Our method for constructing the coefficient functions, $\{u_j(t)\}$, can be motivated by writing $p(D)$ in nested form (*Horner's Algorithm*). Set

$$h_1(D) = 1, \quad h_{k+1}(D) = D h_k(D) + p_k, \ \ k = 1, 2, \ldots n.$$

It is easy to check that $h_{n+1}(D) = p(D)$.

Let w(t) be the solution to the nth order ordinary differential equation,[1]

$$Lw \doteq p(D)w = 0, \quad \text{with } D \doteq d \ \ /dt$$

[1] The solution $w(t)$ is an analytic function of t. It is called the *weighting function* of the differential equation.

such that

$$D^k w(0) = 0, \quad k = 0, 1, 2, \dots n-2 \quad \text{and} \quad D^{n-1} w(0) = 1,$$

then the analytic coefficient functions of the finite representation can be defined by

$$u_1(t) = w(t), \quad u_{k+1}(t) = Du_k(t) + p_k u_1(t), \quad k = 1, 2, \dots n-1.$$

Let

$$\Phi(t) = u_1(t)A^{n-1} + u_2(t)A^{n-2} + \cdots + u_{n-1}(t)A + u_n(t)I.$$

It will be shown that

$$D\Phi = A\Phi \ \ , \Phi(0) = I.$$

Therefore, by the uniqueness theorem of ordinary differential equations, $\Phi(t) = e^{tA}$.

From the definition of the functions $\{u^k(t)\}$, it follows that

$$Du_k(t) - u_{k+1}(t) = -p_k u_1(t), \quad k = 1, 2, \dots n-2.$$

In addition,

$$Du_n(t) = D\{D^{n-1}w(t) + p_1 D^{n-2} w(t) + \cdots + p_{n-1} w(t)\} = Lw(t) - p_n w(t) = -p_n w(t).$$

After substituting the above relations into

$$D\Phi(t) - A\Phi(t) = -u_1(t)A^n + [Du_2(t) - u_1(t)]A^{n-1} + \cdots + [Du_{n-1} - u_n(t)]A + Du_n I,$$

we obtain

$$D\Phi(t)-A\Phi(t) = -w(t)A^n-p_1w(t)A^{n-1}-\cdots-p_{n-1}w(t)A-p_nw(t)I = -w(t)p(A) = 0.$$

The last equality is due to the Cayley-Hamilton theorem.

To show that $\Phi(0) = I$, substitute $t = 0$ into $u_1(t) = w(t)$ and

$$u_{k+1}(t) = D^kw(t) + p_1D^{k-1}w(t) + \cdots + p_kw(t), \quad k = 1, 2, \ldots, n-1.$$

The last step is to invoke the initial data,

$$D^kw(0) = 0, \quad k = 0, 1, 2, \ldots n-2 \quad \text{and} \quad D^{n-1}w(0) = 1,$$

to obtain

$$u_k(0) = 0, \quad k = 0, 1, 2, \ldots n-1 \quad \text{and} \quad u_n(0) = 1,$$

Remark: The proof implies that if $t = 0$, then the Wronskian matrix

$$W(u_1, \ldots, u_n) = I.$$

Remark: If the eigenvalues of A and their multiplicities are known, then it is a straightforward task to compute explicitly the functions $\{u_k(t)\}$.

Example 1. Set $A = [0, 1, 0; 0, 0, 1; -2, 1, 2]$, then $p(\lambda) = \lambda^3-2\lambda^2-\lambda+2$, and

$L[w] = D^3w - 2D^2w - Dw + 2w$. The eigenvalues are $\lambda = 1, -1, 2$,

$$u_1(t) = w(t) = \frac{1}{3}(-\cosh(t) - 2\sinh(t) + e^{2t}),$$

$$u_2(t) = Dw(t) - 2w(t) = \sinh(t),$$

$$u_3(t) = D^2w(t) - 2Dw(t) - w(t) = \frac{1}{3}(4\cosh(t) + 2\sinh(t) - e^{2t}),$$

and

$$e^{tA} = u_1(t)A^2 + u_2(t)A + u_3(t)I.$$

Example 2. Set $A = [2, 2, 1; 1, 3, 1; 1, 2, 2]$, then $p(\lambda) = \lambda^3 - 7\lambda^2 + 11\lambda - 5$, and $L[w] = D^3w - 7D^2w + 11Dw - 5w$. The eigenvalues are $\lambda = 1, 1, 5$,

$$u_1(t) = w(t) = \frac{1}{16}(e^{5t} - (1 + 4t)e^t),$$

$$u_2(t) = Dw(t) - 7w(t) = \frac{1}{16}(-2e^{5t} + (2 + 24t)e^t),$$

$$u_3(t) = D^2w(t) - 7Dw(t) + 11w(t) = \frac{1}{16}(e^{5t} + (15 - 20t)e^t),$$

and

$$e^{tA} = u_1(t)A^2 + u_2(t)A + u_3(t)I.$$

Example 3. Set $A = [1, -1, -1; 1, 1, 0; 3, 0, 1]$, then $p(\lambda) = \lambda^3 - 3\lambda^2 + 7\lambda - 5$, and

$L[w] = D^3w - 3D^2w + 7Dw - 5w$. The eigenvalues are $\lambda = 1, 1 \pm 2i$.

$$u_1(t) = w(t) = \frac{1}{4}e^t(1 - \cos(2t),$$

$$u_2(t) = Dw(t) - 3w(t) = \frac{1}{4}e^t(-2 + 2\cos(2t) + 2\sin(2t)),$$

$$u_3(t) = D^2w(t) - 3Dw(t) + 7w(t) = \frac{1}{4}e^t(5 - \cos(2t) - 2\sin(2t)),$$

and

$$e^{tA} = u_1(t)A^2 + u_2(t)A + u_3(t)I.$$

Example 4. Set $A = [1, 1, 1; 2, 1, -1; -3, 2, 4]$, then $p(\lambda) = \lambda^3 - 6\lambda^2 + 12\lambda - 8$, and

$L[w] = D^3 w - 6D^2 w + 12Dw - 8w$. The eigenvalues are $\lambda = 2, 2, 2$.

$$u_1(t) = w(t) = \frac{1}{2} t^2 e^{2t},$$

$$u_2(t) = Dw(t) - 6w(t) = \frac{1}{2} e^{2t}(t - 2t^2),$$

$$u_3(t) = D^2 w(t) - 6Dw(t) + 12w(t) = \frac{1}{2} e^{2t}(1 - 2t + 2t^2),$$

and

$$e^{tA} = u_1(t)A^2 + u_2(t)A + u_3(t)I.$$