Errata: (*)

(2.13.5) Show that the quad. part of the cubic system

$$\dot{x} = y + x^2 - x^3 + xy^2 - y^3$$
$$\dot{y} = x^2 - 2xy + x^3 - x^2 y$$

Can be reduced to normal form using the xform.

$$x = y + \begin{pmatrix} 0 \\ -ay_1^2 \end{pmatrix} \quad \left(\begin{array}{l} x_1 = y_1 \\ x_2 = y_2 - ay_1^2 \end{array} \text{ or } \; y_2 = x_2 + ax_1^2 \right)$$

and show that this yields

$$\begin{cases} \dot{x} = y - x^3 + xy^2 - y^3 + \Theta(|x|^4) \\ \dot{y} = x^2 + 3x^3 + x^2 y + \Theta(|x|^4) \end{cases}$$

should be $\underline{(-)}^*$

Then determine xform $x = y + h(y)$ with $h = h_3(y) \in H_3$ that reduces further to

$$\begin{cases} \dot{x} = y + \Theta(|x|^4) \\ \dot{y} = x^2 + 3x^3 - 2x^2 y + \Theta(|x|^4) \end{cases}$$

should be $\underline{(-4)}^*$

$$\dot{y}_1 = (y_2 - ay_1^2) + y_1^2 - y_1^3 + y_1(y_2 - ay_1^2)^2 - (y_2 - ay_1^2)^3$$
$$= y_2 + \underbrace{(1-a)}_{a=1} y_1^2 - y_1^3 + y_1(y_2^2 + y_1^4 - 2y_1^2 y_2) - (y_2^3 - 3y_2^2 y_1^2 + 3y_2 y_1^4 - y_1^6)$$
$$= y_2 + (-y_1^3 + y_1 y_2^2 - y_2^3) + (2y_1^3 y_2 - 3y_1^2 y_2^2) + (y_1^5 + 3y_2 y_1^4) - y_1^6$$

$$\dot{y}_2 - 2y_1 \dot{y}_1 = \cancel{y_1^2 + 3y_1^3 + 2y_1^2(y_2 - y_1^2)}$$
$$y_1^2 - 2y_1(y_2 - y_1^2) + (y_1)^3 - y_1^2(y_2 - y_1^2)$$

$$\dot{y}_2 = 2\cancel{y_1 y_2} + 2y_1(-\cancel{y_1^3} + y_1 y_2^2 - y_2^3) + \cdots + y_1^2 - 2\cancel{y_1 y_2} + 3y_1^3 - y_1^2 y_2 + \cancel{y_1^4}$$
$$= y_1^2 + 3y_1^3 - y_1^2 y_2 + \Theta(|x|^4)$$

The homological equation for $J=\begin{pmatrix}0&1\\0&0\end{pmatrix}$;

$\underline{r=2}$ $X_2 - L_A h_2$: $h_2=\begin{pmatrix}a^1_{20}x_1^2+a^1_{11}x_1x_2+a^1_{02}x_2^2\\ a^2_{20}x_1^2+a^2_{11}x_1x_2+a^2_{02}x_2^2\end{pmatrix}$, $X_2=\begin{pmatrix}A^1_{20}x_1^2+A^1_{11}x_1x_2+A^1_{02}x_2^2\\ A^2_{20}x_1^2+A^2_{11}x_1x_2+A^2_{02}x_2^2\end{pmatrix}$

$X_2-(Dh_2)J\begin{pmatrix}y_1\\y_2\end{pmatrix}+Jh_2 =$ $\quad * \; Jy=\begin{pmatrix}y_2\\0\end{pmatrix}$

$$=\begin{pmatrix}(A^1_{20}+a^2_{20})y_1^2 +(A^1_{11}-2a^1_{20}+a^2_{11})y_1y_2+(A^1_{02}-a^1_{11}+a^2_{02})y_2^2\\ A_{20}y_1^2 \quad +(A^2_{11}-2a^2_{20})y_1y_2 \quad +(A^2_{02}-a^2_{11})y_2^2\end{pmatrix}$$

$(r=3)$ $X_3+Jh_3-(Dh_3)\begin{pmatrix}y_2\\0\end{pmatrix}=$

$$\begin{pmatrix}(A^1_{30}+a^2_{30}\cancel{+3a^1_{30}})x_1^3+(A^1_{21}+a^2_{21}-3a^1_{30})x_1^2x_2+(A^1_{12}+a^2_{12}-2a^1_{21})x_1x_2^2+(A^1_{03}+a^2_{03}-a^1_{12})x_2^2\\ A^2_{30}x_1^3+(A^2_{21}-3a^2_{30})x_1^2x_2+(A^2_{12}-2a^2_{21})x_1x_2^2+(A^2_{03}-a^2_{12})x_2^2\end{pmatrix}$$

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here: $\left.\begin{aligned}\dot{x}_1&=x_2+x_1^2-x_1^3+x_1x_2^2-x_2^3\\ \dot{x}_2&=x_1^2-2x_1x_2+x_1^3-x_1^2x_2\end{aligned}\right\}$ $\quad X_2(v)=\begin{pmatrix}x_1^2\\ x_1^2 \; 2x_1x_2\end{pmatrix}$

$A^1_{20}=1,\ A^2_{20}=1,\ A^2_{11}=-2$

Then, $a^2_{20}=-1$, all other $a^k_{ij}=0$

i.e. $\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}y_1\\y_2-y_1^2\end{pmatrix}$

$\dot{y}_1=(y_2-\cancel{y_1^2})+\cancel{y_1^2}-y_1^3+y_1(y_2-y_1^2)^2-(y_2-y_1^2)^3$

$\quad = y_2 + (-y_1^3+y_1y_2^2-y_2^3)+(-2y_1^3y_2-3y_1^2y_2^2)+(y_1^5+3y_2y_1^4)-y_1^6$

$\dot{x}_2=\dot{y}_2-2y_1\dot{y}_1 = y_1^2-2y_1(y_2-y_1^2)+y_1^3-y_1^2(y_2-y_1^2)$

$\Rightarrow \dot{y}_2 = 2y_1[\cancel{y_2}+(-y_1^3+y_1y_2^2-y_2^3)+\mathcal{O}(|y|^4)]+y_1^2-\cancel{2y_1}y_2+3y_1^3-y_1^2y_2+y_1^4$

$\quad = y_1^2+3y_1^3-y_1^2y_2+[-y_1^4+2y_1^2y_2^2-2y_1y_2^3]+\mathcal{O}(|y|^5)$

$$\dot{y}_1 = y_2 + \left[-y_1^3 + y_1 y_2^2 - y_2^3\right] + \left[-2y_1^3 y_2 - 3y_1^2 y_2^2\right] + \mathcal{O}(|y|^5)$$

$$\dot{y}_2 = y_1^2 + \left[3y_1^3 - y_1^2 y_2\right] + \left[-y_1^4 + 2y_1^2 y_2^2 - 2y_1 y_2^3\right] + \mathcal{O}(|y|^5)$$

Now $\left\{\begin{array}{l} A^1_{30} = -1,\ A^1_{12} = 1,\ A^1_{03} = -1 \\ A^2_{30} = 3,\ A^2_{21} = 1 \end{array}\right\} \left\{\begin{array}{l} A^1_{21} = 0 \\ A^2_{12} = A^2_{03} = 0 \end{array}\right.$

So: $\left\{A^1_{30} + a^2_{30} = 0 \text{ or } A^2_{21} - 3a^2_{30} = 0\right\}$ not possible simultaneously
choose $a^2_{30} = -A^1_{30} = 1$

$A^1_{12} + a^2_{12} - 2a^1_{21} = 0$ while $a^2_{21} = a^2_{12} = 0$ $\rightarrow a^1_{30} = 0$

$A^1_{03} + a^2_{03} - a^1_{12} = 0$ $\Rightarrow a^1_{21} = \frac{1}{2} A^1_{12} = \frac{1}{2}$ $a^1_{03} = 0$ (irrelevant)

$\cancel{A^2_{21} + 3a^2_{30} = 0}$. Let $a^2_{03} = 1,\ a^1_{12} = 0$

Then $h_3 = \begin{pmatrix} \frac{1}{2} x_1^2 x_2 \\ x_1^3 + x_2^3 \end{pmatrix}$ which gives $\hat{X}_3(y)$:

$$\hat{X}_3(y) = X_3 + Jh_3 - Dh_3\begin{pmatrix} y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3y_1^3 - 4y_1^2 y_2 \end{pmatrix}$$

So: let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} z_1^2 z_2 \\ z_1^3 + z_2^3 \end{pmatrix}$

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ z_1^2 + 3z_1^3 - 4z_1^2 z_2 \end{pmatrix} + \mathcal{O}(|z|^4)$$