#12

2.2.4) Use the method of successive approximation to show that if the matrix-valued function $A(t)$ is continuous in $I_0=[-a,a]$ then $\exists\, a>0$ such that the IVP $\dot{\Phi}=A\Phi\,;\ \Phi(0)=\mathbb{I}$ ($\mathbb{I}$ is $n\times n$ identity) has a unique fundamental matrix solution $\Phi(t)$ on $[-a,a]$.

Let $\Phi_0(t)=\mathbb{I}\,;\quad \Phi_{k+1}=\mathbb{I}+\int_0^t A(s)\Phi_k(s)\,ds \quad (*)$.

Since $\|A(t)\|$ continuous on compact set $\Rightarrow \|A(t)\|\leq M \quad \forall t\in[-a_0,a_0]$

Now, for $t\in[-a,a]\subset[-a_0,a_0]$:

$$\Phi_1(t)=\mathbb{I}+\int_0^t A(s)\,ds \Rightarrow \|\Phi_1-\overset{\overset{\mathbb{I}}{\shortparallel}}{\Phi_0}\|\leq\int_0^{|t|}\|A(s)\|ds\leq Ma$$

Assuming $\|\Phi_k-\Phi_{k-1}\|\leq(Ma)^k,\ \forall t\in[-a,a]$ (induction hypothesis)

follows: $\|\Phi_{k+1}-\Phi_k\|\leq\int_0^{\|t\|}\|A(s)\|(\|\Phi_k-\Phi_{k-1}\|)\,ds\leq M(Ma)^k a=(Ma)^{k+1}$

$\Rightarrow$ True for all $k=1,2,\ldots$ Let $\alpha=Ma$. Then if $m>n>N$:

$$\|\Phi_m-\Phi_n\|\leq\sum_{k=n}^{m-1}\|\Phi_{k+1}-\Phi_k\|\leq\sum_{k=N}^{\infty}\|\Phi_{k+1}-\Phi_k\|\leq\sum_{k=N}^{\infty}(Ma)^k=\frac{\alpha^N}{1-\alpha}\xrightarrow[\alpha<1]{N\to\infty}0$$

$\Rightarrow\{\Phi_m\}_{m=0}^{\infty}$ is a Cauchy sequence $\Rightarrow$ Converges to continuous function uniformly on $t\in[-a,a]$ where $a=\min\left(\frac{1}{M},a_0\right)$ $\left\{\alpha<1 \Leftrightarrow a<\frac{1}{M}\right.$

Taking limit on both sides of $(*)$:

$$\Phi=I+\int_0^t A\Phi\,ds$$

Φ,A continuous $\Rightarrow$ rhs diff/ble $\Rightarrow \left\{\dfrac{\dot{\Phi}=A\Phi \text{ with } \Phi(0)=I}{t\in[-a,a]}\right.$

For uniqueness:

Let Φ,Ψ solutions ($\Rightarrow$ Continuous on $[-a,a]$)

$\Rightarrow \|\Phi-\Psi\|$ has maximum on $[-a,a]$. But then $t_1\in[-a,a]$:

$$\max_{t\in[-a,a]}\|\Phi(t)-\Psi(t)\| = \|\Phi(t_1)-\Psi(t_1)\| = \left\|\int_0^{t_1}A(s)(\Phi(s)-\Psi(s))\,ds\right\|$$

$$\leq M\int_0^{|t_1|}\|\Phi(s)-\Psi(s)\|\leq Ma\,\|\Phi(t_1)-\Psi(t_1)\|$$

But $Ma<1$ $\Rightarrow$ inequality can only be true if $\|\Phi(t_1)-\Psi(t_1)\|=0$

$\Rightarrow \Phi(t)\equiv\Psi(t)$ (max. difference vanishes)

#13

2.3.4 Let E be an open subset of $\mathbb{R}^n$ containing y_0. Use the method of successive approximations and Gronwall's lemma to show that if $A(t,y)$ is continuous on $[-a_0,a_0]\times E$ then there exist an $a>0$ and a $\delta>0$ such that for all $y\in N_\delta(y_0)$ the IVP

$$\dot{\Phi}=A(t,y)\Phi \;;\; \Phi(0,y)=I \qquad (*)$$

has a unique solution $\Phi(t,y)$ continuous on $[-a,a]\times N_\delta(y_0)$

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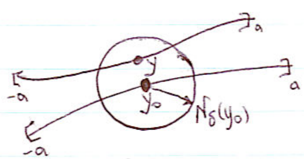
Note: Since $A(t,y)$ is continuous for $t\in I\equiv[-a_0,a_0]$ we have already shown that there exists a & such that $(*)$ has a unique solution. What remains to show is that the solution depends continuously on the parameter y.



Since

$$\underbrace{\Phi(t,y)}_{\Phi_y}-\overbrace{\Phi(t,y_0)}^{\Phi_{y_0}}=\int_0^t\big(A(s,y)\Phi(s,y)-A(s,y_0)\Phi(s,y_0)\big)ds$$

$$=\int_0^t\Big[\big(A(s,y)-A(s,y_0)\big)\Phi(s,y_0)+A(s,y)\big(\Phi(s,y)-\Phi(s,y_0)\big)\Big]ds$$

$$\Rightarrow \|\Phi_y-\Phi_{y_0}\|\le\int_0^{|t|}\|A(s,y)-A(s,y_0)\|\,\|\Phi(s,y_0)\|\,ds+\int_0^{|t|}\|A(s,y)\|\,\|\Phi(s,y)-\Phi(s,y_0)\|\,ds$$

$$\le \varepsilon b a+M\int_0^{|t|}\|\Phi_y-\Phi_{y_0}\|\,ds$$

where we used: (i) $\|A(t,y)\|\le M$ } for $(t,y)\in[-a,a]\times N_\delta(y_0)$
(ii) $\|\Phi(t,y_0)\|\le b$ } ~~for~~ *uniform continuity of $A\to$ uniform ~~continuity~~
(iii) $\forall\varepsilon>0\ \exists\delta$: $\max_{t\in[-a,a]}\|A(t,y)-A(t,y_0)\|\le\varepsilon$ $\forall y\in N_\delta(y_0)$

by Gronwall's lemma $\Longrightarrow$ $\|\Phi(t,y)-\Phi(t,y_0)\|\le\varepsilon b a\, e^{M|t|}$

which establishes the (uniform) continuity of $\Phi(t,y)$ w.r.t y in $N_\delta(y_0)$ (continuity wrt t already proven).

(*continuity is uniform since $t\in[-a,a]$, a closed interval (int))