466 '07 (E.A. Coutsias)-Final

due: Thursday, December 13, 2007

December 9, 2007

1. Find the normal modes of a membrane having the shape of a circular ring if its outer boundary is rigidly fixed and its inner boundary is elastically fixed. Thus you must find the eigenfunctions of

$$\begin{cases} \triangle u + k^2 u = 0 \ , \quad a < r < b \\ u(b, \theta) = 0 \\ u_r(a, \theta) - h u(a, \theta) = 0 \end{cases}$$

   Here h is a positive constant; <u>k must be found</u>.
   (Hint: Write

$$u(r, \theta) = \frac{a_0(r)}{2} + \sum_1^\infty \left(a_n(r) \cos n\theta + b_n(r) \sin n\theta \right) \ .$$

   Derive and solve equations for the $a_k(r)$, $b_k(r)$. Also <u>k is not given</u>; rather, $k = \omega / c$ is the <u>unknown</u> vibration "frequency", <u>the eigenvalue</u>).

2. Find the disturbance due to a harmonic point source in an infinitely long acoustic waveguide with hard walls. Thus, solve

$$\triangle u + k^2 u = \delta(x - \xi)\delta(y - \eta) \quad , \quad 0 < x < \infty \ , \ 0 < y < b$$
$$u \ \text{ outgoing at } x = \pm\infty \quad , \quad u = 0 \ \text{ on } \ y = 0 \ \text{ and } y = b \ .$$

   Here we are solving

$$v_{xx} + v_{yy} - \frac{1}{c^2} v_{tt} = \delta(x - \xi)\delta(y - \eta) e^{i\omega t}$$

   with

$$v(x, y, t) = u(x, y) e^{i\omega t} \ , \ k = \frac{\omega}{c} \ .$$

(Hint: let

$$u(x,y) = \sum_1^\infty u_n(x) \sin \frac{n\pi y}{b} \ ;$$

multiply d.e. by $\sin(n\pi y/b)$ and integrate over y from 0 to b. Then solve the resulting problem for the $u_n(x)$ either by Fourier transform or by constructing Green's function directly.)

3. Consider

$$\begin{aligned} u_{xx} + u_{yy} = 0 \quad &, \quad 0 < x < \infty \ , \ 0 < y < 1 \\ u(x,y) \to 0 \text{ as } x \to \infty \quad &, \quad u(0,y) = 0 \ , \quad 0 < y < 1 \\ u(x,1) = 0 \ , \ u(x,0) = f(x) \ , \text{ given.} \quad &, \quad 0 < x < \infty \end{aligned}$$

Find two different representations of the solution

   (a) By expanding in y-eigenfunctions.
   (b) By using an appropriate transform in x.

4. The equation describing the electric potential in a cable is

$$V_{xx} = LCV_{tt} + (RC + LG)V_t + RGV \tag{1}$$

where	$L =$	inductance per unit length
	$C =$	capacitance to ground per unit length
	$R =$	resistance per unit length
and	$G =$	leakage to ground per unit length.

Find the solution of the initial value problem

$$V(x,0) = f(x) \ , \ V_t(x,0) = 0 \ , \ -\infty < x < \infty \ .$$

The form of the equation suggests trying as a solution

$$V(x,t) = e^{-\gamma t} F(\alpha x + \beta t) \ ,$$

where α, β, γ are constants and F is an arbitrary function. Show that such a solution is possible if and only if $RC = LG$.

In fact, if $RC = LG$, the general solution to eq.(1) is

$$V(x,t) = e^{-kt} \left[F\left(x - \frac{t}{\sqrt{LC}}\right) + H\left(x + \frac{t}{\sqrt{LC}}\right) \right]$$

where $k = R/L = G/C$, and F, H are arbitrary functions. s represents a combination of waves moving with speed $1/\sqrt{LC}$ and preserving their shape but becoming attenuated (damped) in time. The (telegraph) cable is seen to transmit in this way only if $RC = LG$ and it is then said to be *distortionless*. If $RC \neq LG$ the signal suffers both attenuation and distortion.

5. The (wavefunction) $\psi(x)$ for the Quantum Simple Harmonic Oscillator satisfies the dimensionless, stationary Schrödinger equation

$$\frac{d^2\psi}{dx^2} + (2\lambda + 1 - x^2)\psi = 0 \ .$$

Use the transformation

$$\psi(x) = exp\left(-\frac{x^2}{2}\right) f(x)$$

to prove that

$$\frac{d^2 f}{dx^2} - 2x\frac{df}{dx} + 2\lambda f = 0 \ .$$

Construct two linearly independent solutions of this (*Hermite's*) equation in the neighborhood of $x = 0$. Give the radius of convergence of the series obtained and show that for special values of λ one of the solutions is a polynomial. Find these special values (*eigenvalues*) of λ and calculate the corresponding eigenfunctions.