466 '07 (E.A. Coutsias)-HOMEWORK 8

due: Tuesday, November 13, 2007

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1. Show that the differential equation

$$z^2 w'' + (1 - 2\alpha) z w' + \left(\beta^2 \gamma^2 z^{2\gamma} + \alpha^2 - \nu^2 \gamma^2\right) w = 0$$

has Bessel function solutions

$$w = A z^\alpha J_\nu (\beta z^\gamma) + B z^\alpha Y_\nu (\beta z^\gamma)$$

if ν is an integer or zero. Hence solve

$$w'' - zw = 0 \quad (1)$$

$$zw'' - w' + 4z^3 w = 0 \quad (2)$$

(*Note*: The solutions to the first equation are called *Airy functions* and are of importance in the theory of diffraction and wave propagation).

2. Show that the Wronskian of $J_\nu(z)$ and $J_{-\nu}(z)$ is $-2\sin(\pi\nu)/(\pi z)$. What is the Wronskian of $J_\nu(z)$ and $Y_\nu(z)$?
(Recall $\Gamma(1-\nu)\Gamma(\nu) = \pi \csc \pi\nu$).

3. Deduce from the recursion relations developed in class that

$$\frac{d}{dz}\left[z^n J_n(z)\right] = z^n J_{n-1}(z)$$

$$\frac{d}{dz}\left[z^{-n} J_n(z)\right] = -z^{-n} J_{n+1}(z) \ .$$

Use Rolle's theorem to prove that one zero of $J_{n+1}(x)$ lies between successive zeros of $J_n(x)$ (x is real).

4. Use the relationship

$$Y_\nu(z) = \frac{1}{\sin \nu\pi}\left[\cos \nu\pi J_\nu(z) - J_{-\nu}(z)\right]$$

to prove that $Y_\nu(z)$ satisfies the recursion relations

$$\begin{aligned} 2Y'_\nu &= Y_{\nu-1} - Y_{\nu+1} \\ Y_{\nu+1} &= \frac{\nu}{z}Y_\nu - Y'_\nu \ . \end{aligned}$$

5. According to the elementary theory of nuclear reactors, the steady state neutron density u in a reactor is given by

$$\nabla^2 u + B^2 u = 0$$

where $B > 0$ is a physical constant. For a spherical reactor of radius R determine solutions independent of θ and ϕ. Determine the smallest value of BR for which $u = 0$ when $r = R$.
(*Hint:* Use the results of problem (1) to relate the solution to $J_{\pm 1/2}$; then use the series expression for Bessel functions to establish the formulas for $J_{\pm 1/2}$ given in the notes, p.16.3)

6. (a) Show that for Legendre polynomials $P_n(x)$:

$$\begin{aligned} P_{2n+1}(0) &= 0 \\ P_{2n}(0) &= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \ . \end{aligned}$$

(b) Use the recurrence relations on p.17.7 of the notes to deduce that

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \ .$$

(c) Use part (b) to show that

$$\int_0^1 P_n(x)dx = \begin{cases} 1 & \text{if } \ n = 0 \\ 0 & \text{if } \ n = 2m \ , \ m > 0 \\ \dfrac{(-1)^m (2m)!}{2^{2m+1} m!(m+1)!} & \text{if } \ n = 2m+1 \end{cases}$$