466 '06 (E.A. Coutsias)-HOMEWORK 6

due: Friday, October 26, 2007

October 24, 2007

1. (a) Given

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \ , \ \Gamma(\beta) = \int_0^\infty y^{\beta-1} e^{-y} dy$$

show that

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= 2\Gamma(\alpha+\beta) \int_0^{\pi/2} \cos^{2\alpha-1}\theta \sin^{2\beta-1}\theta d\theta \\ &= \Gamma(\alpha+\beta) \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \ . \end{aligned}$$

(b) Use the result from problem 4.3c, namely that

$$\int_{-\infty}^\infty \frac{e^{\alpha t}}{1+e^t} dt = \frac{\pi}{\sin \pi\alpha}$$

to show that

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} \ .$$

(*Hint:* Use polar coordinates.)

2. The Laplace transform of $f(t)$ is

$$F(s) = \frac{1}{(s-a)^{\nu+1}} \ , Re(\nu) > -1 \ .$$

Find $f(t)$ by contour integration. Here, we want a real answer for $f(t)$ when ν is real so the branch should be chosen so that $F(s)$ is real when s is real, $s > a$. a is a real constant.
(*Hint:* See the posted notes; the inversion using the contour integral formula is straightforward for $-1 < Re(\nu) < 0$, while the extension to positive real parts can be done by using integration by parts to reduce the exponent to the case $-1 < Re(\nu) < 0$.)

3. Use Laplace transforms to solve the following problem for $u(t)$:

$$\frac{du}{dt} + \int_0^t u(x)dx = H(t) = \begin{cases} 1 & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases} \quad , \; u(0) = 2 \; .$$

4. $f(t)$ is periodic with period T, i.e. $f(t + nT) = f(t)$, for n integer,

$$g(t) = \begin{cases} f(t) & , \quad 0 < t < T \\ 0 & , \quad t > T \end{cases} \quad .$$

Show that

$$F(s) = \frac{G(s)}{1 - e^{-sT}} \; , \; s > 0 \; .$$

Use this result to show that if $f(t) = |\sin \omega t|$, then

$$F(s) = \frac{\omega}{s^2 + \omega^2} \coth\left(\frac{\pi s}{2\omega}\right) \; .$$

5. *Abel's* integral equation for the function $\phi(t)$, $t > 0$ is

$$\int_0^t \phi(\tau)\,(t - \tau)^{-\alpha}\, d\tau = f(t) \; , \; t > 0 \; ,$$

where α is a real constant, $0 < \alpha < 1$, and $f(t)$ is given. Solve for $\phi(t)$ using Laplace transforms.

6. Given that

$$J_0(at) = \frac{1}{\pi} \int_0^\pi \cos{(at \sin\theta)} d\theta$$

show that the Laplace transform of $J_0(at)$ is $(s^2 + a^2)^{-1/2}$. Hence prove that

$$\int_0^t J_0(\tau) J_0(t - \tau) d\tau = \sin t \; .$$

7. The equation

$$\frac{\partial^2 u}{\partial x^2} - k^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \; ,$$

where k and c are (positive) physical constants, governs the motion of a string on an elastic foundation. Consider the case of a semi-infinite string, initially at rest with zero displacement and subject to the boundary condition $u = \sin \omega t$, $x = 0$, $t > 0$. Using a Laplace transform method, find the solution in the transformed domain. Outline the main steps involved in the inversion for $\omega > kc$.