466 '07 (E.A. Coutsias)-HOMEWORK 3

due: Thursday, September 13, 2007

September 11, 2007

1. Consider the power series

$$P(z) = \frac{1}{1!} + \frac{1}{2!}z + \frac{1}{3!}z^2 + \cdots = \frac{e^z - 1}{z} \ .$$

   The reciprocal series

$$P^{-1}(z) = \frac{z}{e^z - 1} =: \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

   defines the Bernoulli numbers, B_n. Using the Wronski fromulas show that

$$B_0 = 1 \ , \ B_1 = -\frac{1}{2} \ , \ B_2 = \frac{1}{6} \ , \ B_3 = 0 \ , \ B_4 = -\frac{1}{30} \ .$$

2. Consider the series $A(z) = \sum_0^\infty a_n z^n$ where a_n satisfy the recurrence relation $a_{n+2} = a_{n+1} + 2a_n$, with $a_0 = a_1 = 1$ (i.e. $a_2 = 3$, $a_3 = 5$, $a_4 = 11$,... etc).
   (i) By using the substitution $a_n = r^n$, show that

$$a_n = c_1(-1)^n + c_2 2^n$$

   where $c - 1$, c_2 are constants. Find c_1 and c_2.
   (ii) Use the Wronski formulas to show that

$$A^{-1}(z) = 1 - z - 2z^2 \ .$$

   (iii) What is the radius of convergence of A?
   (*Hint: relate A^{-1} to A. Where does A^{-1} vanish?*)

3. Use the binomial series

$$Q_\alpha(z) = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \binom{\alpha}{3} z^3 + \cdots = (1+z)^\alpha$$

and the property $Q_\alpha(z)Q_\beta(z) = Q_{\alpha+\beta}(z)$ to derive the identity (Vandermonde formula)

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} = \binom{\alpha+\beta}{n}$$

Hint: use the Cauchy product formula for series:

$$\left(\sum_0^\infty a_n z^n\right)\left(\sum_0^\infty b_n z^n\right) = \sum_0^\infty c_n z^n$$
$$\Rightarrow \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

4. Show that the two Laurent expansions

$$L_1(z) = -\left(\frac{1}{z} + 1 + z + z^2 + \cdots\right) \ , \ 0 < |z| < 1$$

and

$$L_2(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots \ , \ |z| > 1$$

are analytic continuations of each other.
(*Hint: show that, although they are valid in nonoverlapping domains, they both sum to the "same" function - which can thus be extended to a single function, valid in both domains. What is that function?*).

5. Given that

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} \ ; \ x \text{ real.}$$

Show that the analytic continuation of $\tan^{-1} x$ to complex values ($x \to z$) is

$$\tan^{-1} z = \int_0^z \frac{dt}{1+t^2} = \frac{1}{2i}\int_0^z \left(\frac{1}{t-i} - \frac{1}{t+i}\right) dt$$
$$= \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) .$$

(*Hint: show that* $\tan^{-1} z$ *reduces to* $\tan^{-1} x$ *for* z *real.*)

6. How is the half-plane $Re(z) > 1$ mapped by $w = z^2$? Discuss also the image of $Re(z) > -1$.

7. Determine a branch of $(1-z^2)^{1/2}$ which takes on the value $+1$ at $z = 0$. Find the values on this branch at $z = +2$, $z = -2$, as each point is approached along a path in the upper and lower half-planes (i.e. four values are requested).
(*Hint: put the cuts* $|x| > 1$, $y = 0$.)

8. What part of the $z-$plane corresponds to the interior of the unit circle in the $w-$plane if
$$w = \frac{z-i}{z+i} \ ?$$

Note: The Wronski formulas for series inversion

Given the (formal) power series $A(z) = \sum_0^\infty a_n z^n$ with $a_0 \neq 0$, the series for the reciprocal, A^{-1} is given by $A^{-1} =: B =: \sum_0^\infty b_n z^n$ where the reciprocal coefficients b_k are found by requiring $AB = 1$. Applying the Cauchy product formula and solving for the b_k successively, it can be shown that

$$b_0 = \frac{1}{a_0} \ , \ b_1 = -\frac{a_1}{a_0^2} \ , \ \cdots \ , \ b_k = \frac{(-1)^k}{a_0^{k+1}} \begin{vmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ a_0 & a_1 & \cdots & a_{k-2} & a_{k-1} \\ 0 & a_0 & \cdots & a_{k-3} & a_{k-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 & a_1 \end{vmatrix} \ .$$