466 '07-HOMEWORK 2

due: Thursday, September 6, 2007

September 5, 2007

1. Evaluate each of the following contour integrals:

   (a) $\quad I_1 = \int_{\mathcal{C}} (z-1)dz$ where $\mathcal{C}$ is the straight line from $z = 1$ to $z = i$.

   (b) $\quad I_2 = \int_{-i}^{i} \left(z^2 + iy^2\right) dz$ on the right half of the unit circle $|z| = 1$, traversed in an anticlockwise direction .

   (c) $\quad I_3 = \int_{\mathcal{C}} \frac{(z+2)}{z} dz$ on $z = 2e^{i\theta}$, $0 \leq \theta \leq 2\pi$,

   where $\mathcal{C}$ is a simple closed (counterclockwise) contour, enclosing the origin.

2. The Bessel function $I_0(t)$ can be defined by

$$I_0(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n}}{(n!)^2} \ .$$

   Show that

$$\frac{(t/2)^{2n}}{(n!)^2} = \frac{(t/2)^n}{(n!)} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{e^{zt/2}}{z^{n+1}} dz$$

   where $\mathcal{C}$ is a simple closed contour including the origin. Using this result show that

$$I_0(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{e^{\frac{t}{2}\left(z+\frac{1}{z}\right)}}{z} dz = \frac{1}{2\pi} \int_0^{2\pi} e^{t\cos\theta} d\theta \ .$$

3. In solving the problem of torsional oscillations in a jet engine, the following contour integral arises for the angular velocity ω of the compressor

$$\omega = \frac{K}{2\pi i} \int_{\mathcal{C}} \frac{e^{zt}}{z^2\left(z^2+a^2\right)} dz$$

where

$$K = \frac{Tt}{I_1 + I_2}\ ,\ a^2 = \tau\left(\frac{I_1+I_2}{I_1 I_2}\right)$$

with:

	T	the turbine torque
	τ	the torque to twist the coupling shaft through 1 rad
with:	I_1	the moment of inertia of the turbine
	I_2	the moment of inertia of the compressor
	$\mathcal{C}$	arbitrary simple closed curve enclosing points $z = 0,\ \pm ia$.

Show that the angular velocity ω can be considered to have two parts, one which increases linearly with time, and one which represents an oscillation of angular frequency a.

4. If $f(z)$ is an entire analytic function such that $\mathcal{R}e\left[f(z)\right] \leq M$ for all z (with M a constant), show that $f(z)$ is constant.
(Hint: apply Liouville's thm. to the entire function $e^{f(z)}$).

5. Let $f(z)$ be an entire function which is real on the real axis and imaginary on the imaginary axis. Prove that f is an odd function.
Hint: apply the Schwarz reflection principle to the function $if(iz)$.

6. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{k^3}{3^k} z^k\ .$$

Compute each of the following and give reasons for your steps

(a) $f^{(6)}(0)$

(b) $\displaystyle\int_{|z|=1} \frac{f(z)}{z^5} dz$

(c) $\displaystyle\int_{|z|=1} e^z f(z) dz$

(d) $\displaystyle\int_{|z|=1} \frac{f(z)\sin z}{z^2} dz$

7. Find the radius of convergence of the series

$$(a) \qquad z + \frac{(a-b)z^2}{2!} + \frac{(a-b)(a-2b)z^3}{3!} + \dots$$

$$(b) \qquad \sum_{n=1}^{\infty} \frac{n}{2^n}(z-i)^n$$

$$(c) \qquad \sum_{n=2}^{\infty} \left(\frac{n+1}{2+n}\right)(z-2i)^n$$

8. The fundamental theorem of algebra implies that any polynomial can be factorized

$$P(z) = C(z-z_1)\dots(z-z_n)$$

where z_1 , ... , z_n are the roots of $P(z)$. Use the result

$$\frac{(fg\dots jk)'}{fg\dots jk)} = \frac{f'}{f} + \frac{g'}{g} \dots + \frac{k'}{k}$$

to show that the number of roots of $P(z)$ inside any contour $\mathcal{C}$ is given by

$$N = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{P'(z)}{P(z)} dz \ .$$

(*Note: the factors above are not necessarily distinct; N counts the roots interior to the contour $\mathcal{C}$ with their multiplicities. It is assumed here that no roots are actually situated on $\mathcal{C}$.*)