466 '07 (E.A. Coutsias)-Problems

To be discussed in problems sessions, 5-6pm, Nov. 29&Dec. 6

November 21, 2007

1. Solve

$$u_{xx} + u_{yy} = F(x,y) \quad , \quad \text{for } \ 0 < x < a \ , \ 0 < y < b$$
$$\text{subject to} \qquad u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0 \ .$$

2. Solve

$$u_t - u_{xx} = F(x,t) \quad , \quad \text{for } \ 0 < x < 1 \ , \ 0 < t$$
$$\text{subject to} \qquad u(0,t) = u(1,t) = 0 \ , \ u(x,0) = f(x) \ .$$

3. Solve

$$u_{xx} - \frac{1}{c^2} u_{tt} = \delta(x-\xi) e^{-i\omega t} \ \text{ on } \ -\infty < x < \infty$$

where $\delta(x-\xi)$ is the Dirac delta function. Thus $u(x,t)$ represents the disturbance due to a source at the point $x = \xi$ emitting a signal of angular frequency ω.

4. Consider the problem of capillary waves on the free surface of a liquid confined in a circular basin of infinite depth. It is known that the equations of motions describing this phenomenon are

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = 0$$

in the body of the liquid

$$\phi_r = 0 \text{ on the fixed walls of the container}$$

$$\left.\begin{array}{l} \phi_t - g\zeta + \frac{T}{\rho}\left(\zeta_{rr} + \frac{1}{r}\zeta_r + \frac{1}{r^2}\zeta_{\theta\theta}\right) = 0 \\ \zeta_t = -\phi_z \end{array}\right\} \ \text{on the free surface}$$

$$\phi \to 0 \quad \text{as} \quad z \to -\infty \ .$$

Here, ϕ is the velocity potential, ζ is the elevation of the free surface (i.e., $z = \zeta(r, \theta, t)$ is the elevation of the free surface, giving the elevation above the equilibrium plane, $z = 0$), g is the acceleration of gravity, T is the surface tension and ρ is the density. Find the shapes of the allowed capillary waves and their frequencies.

5. Find the normal modes of a clamped vibrating rectangular membrane. Thus, you must find the eigenfunctions of

$$u_{xx} + u_{yy} = \frac{1}{c^2} \quad , \quad 0 < x < a \ , \ 0 < y < b$$
$$\text{subject to} \qquad u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \ .$$

What is the frequency of each normal mode? Show that the number $M_{<N}$ of normal modes whose frequency is less than N is approximately equal to the area of a quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{4N^2}{c^2} \ .$$

Thus, show that the number is roughly equal to

$$M_{<N} \approx \frac{\pi a b N^2}{c^2} \ .$$

6. Find the Green's functions to solve the following problems:

$$u'' + k^2 u = f(x) \quad , \quad u(0) = 0 \ , \ u(1) = 0 \ . \tag{1}$$
$$u'' + k^2 u = f(x) \quad , \quad u(0) = u(1) \ , \ u'(0) = -u'(1) \ . \tag{2}$$
$$u'' + \frac{1}{x} u' + \left(k^2 - \frac{16}{x^2}\right) u = f(x) \quad , \quad u \text{ bounded at } \ x = 0 \ , \ u(a) = 0 \ , \ a > 0 \tag{3}$$

In each case state for what real values of k the Green's function does or does not exist.

7. Find the formula for the vibrations of a gas in a spherical vessel, produced by small oscillations of its wall begun at time $t = 0$ if the velocity of the wall is radial and equal to $f(\theta)\cos\omega t$. That is, solve

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) \qquad r < R\ ,\ 0 \le \theta \le \pi\ ,\ t > 0$$

$$u_r(R,\theta,t) = f(\theta)\cos\omega t \quad , \quad u(r,\theta,0) = u_t(r,\theta,0) = 0\ .$$

Hint: separate variables to show that the azimuthal eigenfunction are $\Theta_n(\theta) = P_n(\cos\theta)$, where $P_n(x)$ are the Legendre polynomials, and the radial eigenfunctions are spherical Bessel functions

$$R_n(r) = \frac{1}{\sqrt{kr}}J_{n+\frac{1}{2}}(kr)$$

with k determined by the condition

$$J_{n+\frac{1}{2}}(ka) = 0\ .$$

Then let

$$u(r,\theta,t) = \sum_{n=0}^{\infty} u_n(r,t)P_n(\cos\theta)\ ,$$

and find the equation satisfied by $u_n(r,t)$.