466 '07 (E.A. Coutsias)-Midterm 2

due: Tuesday, December 4, 2007

November 29, 2007

1. Consider the ODE
$$\frac{d^2u}{dz^2} + \left(2 - \frac{2}{z^2}\right) u = 0 \ .$$
Find the first two (nonzero) terms in the series for each of two linearly independent solutions near $z = 0$.

2. Consider the ODE
$$x\frac{d^2u}{dx^2} + (2 - x)\frac{du}{dx} + \lambda u = 0 \ .$$
Find a series expansion for the solution which is regular in the neighborhood of $x = 0$ (find the general recurrence relation). What is the radius of convergence of the series? For certain values of λ (called *eigenvalues*) your series should reduce to polynomials (called *eigenfunctions*). What are the eigenvalues λ_n? The corresponding eigenfunctions (call them $L_n(x)$) are orthogonal on $0 \leq x < \infty$ with respect to an appropriate weight function. Derive this orthogonality relation. Find $L_0(x)$, $L_1(x)$ and $L_2(x)$.

3. The inner boundary of a circular annulus of radii 1 and $R > 1$ is maintained at temperature zero while the outer boundary is maintained at temerature $4 + \cos 6\theta + \sin\theta$. Find the steady state temperature in the annulus. Thus you must solve
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad , \quad 1 < r < R$$
$$u(1,\theta) = 0 \quad , \quad u(R,\theta) = 4 + \cos 6\theta + \sin\theta$$
(you must evaluate all expansion coefficients).

4. Consider an infinitely long channel of unit width with acoustically hard walls. We wish to propagate time-harmonic waves of fixed (angular) frequency ω from left to right along the channel. The waves are governed by the wave equation

$$\triangle u = \frac{1}{c^2} u_{tt} \ ,$$

where c is the (fixed) speed of wave propagation in the channel. Thus you are being asked to find all possible time-harmonic wave functions $u(x, y, t)$ which satisfy

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt} \ , \ \ u(x, 0, t) = u(x, 1, t) = 0 \ .$$

If we now keep c fixed but vary ω, can we propagate waves of arbitrary ω along this channel from left to right?

5. Find the first 3 terms in each of two independent series solutions for the following equations about the point indicated

$$z \frac{d^2 w}{dz^2} + \frac{dw}{dz} - 4zw = 0 \quad , \quad z = 0$$

$$z^3 \frac{d^2 w}{dz^2} + (z^2 - 1) \frac{dw}{dz} + zw = 0 \quad , \quad z = \infty \ .$$