① Suppose there is an epidemic in which every month one third of those who are well become sick, and one quarter of those who are sick become well, while another quarter of those who are sick become dead. Find the steady state for the corresponding Markov process.

(Hint: compare to the formulation of problem 5.3.5, p. 272)

Let $z_k = \begin{pmatrix} d_k \\ s_k \\ h_k \end{pmatrix}$; then $z_{k+1} = \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 1/2 & 1/3 \\ 0 & 1/4 & 2/3 \end{pmatrix} z_k \equiv A z_k$

and $z_k = A^k z_0$; so, diagonalize: $A = V \Lambda V^{-1}$

Now: $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1/4 & 0 \\ 0 & 1/2-\lambda & 1/3 \\ 0 & 1/4 & 2/3-\lambda \end{vmatrix} = (1-\lambda)\left\{\left(\frac{1}{2}-\lambda\right)\left(\frac{2}{3}-\lambda\right) - \frac{1}{12}\right\} = 0$

$\Rightarrow (1-\lambda)\left\{\lambda^2 - \frac{7}{6}\lambda + \frac{1}{4}\right\} = 0$

$\lambda_\pm = \left[7/6 \pm \sqrt{49/36 - 1}\right]/2 = \frac{7}{12} \pm \frac{\sqrt{13}}{12}$

i.e. the spectrum of A is $sp\{A\} = \left\{1, \frac{7+\sqrt{13}}{12}, \frac{7-\sqrt{13}}{12}\right\}$

Clearly $\lambda_1 = 1$ dominates, while $|\lambda_{2,3}| = \left|\left(\frac{7 \pm \sqrt{13}}{12}\right)\right| < 1$.

Now $(A - I)\bar{x} = \begin{pmatrix} 0 & 1/4 & 0 \\ 0 & -1/2 & 1/3 \\ 0 & 1/4 & -1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = 1,\ x_2 = x_3 = 0;\ \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Let $\lambda_{2,3}$ have corresponding eigenvectors ($\bar{v}_2, \bar{v}_3$).

Then $\bar{z}_0 = c_1 \bar{v}_1 + c_2 \bar{v}_2 + c_3 \bar{v}_3$

and $\bar{z}_k = c_1 \bar{v}_1 + c_2 \bar{v}_2 \lambda_2^k + c_3 \bar{v}_3 \lambda_3^k$

$\xrightarrow{k \to \infty} c_1 \bar{v}_1$

$\therefore$ The steady state is $\bar{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and dust to dust...

(2) Find the maximum and minimum values of:

$$R(x) = \frac{x_1^2 - 2x_1x_2 + 2x_2^2 + x_3^2}{x_1^2 + x_2^2 + 4x_3^2}$$

(Hint: apply Rayleigh quotient thm.)

Let $(y_1, y_2, y_3) = (x_1, x_2, 2x_3)$; then

$$R(x) = \hat{R}(y) = \frac{y_1^2 - 2y_1y_2 + 2y_2^2 + \frac{1}{4}y_3^2}{y_1^2 + y_2^2 + y_3^2}$$

Then $\min\limits_{\vec{x}\neq 0} R = \min\limits_{\hat{y}\neq 0} \hat{R} = \lambda_1$; $\max\limits_{x\neq 0} R = \lambda_3$

where λ_1, λ_3 are the least & greatest evals of A, with

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix};$$ we have $|A - \lambda I| = (\frac{1}{4} - \lambda)\{(1-\lambda)(2-\lambda) - 1\} = 0$

$\Rightarrow (\frac{1}{4} - \lambda)[\lambda^2 - 3\lambda + 1]$

$\lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$; clearly

$\lambda_+ = \frac{3+\sqrt{5}}{2} > \cancel{\frac{1}{4}}$ $\lambda_- = \frac{3-\sqrt{5}}{2} > \frac{1}{4}$

so:

$$\left.\begin{aligned} \min R(x) &= \frac{1}{4} \\ \max R(x) &= \frac{3+\sqrt{5}}{2} \end{aligned}\right\}$$

(3) Determine the dimension and a basis for each

⟨a⟩ of the subspaces $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A^T)$, $\mathcal{R}(A^T)$ for

$$A = \begin{pmatrix} 2 & 1 & 4 & -2 \\ 1 & 3 & 1 & 1 \\ 0 & 5 & -2 & 4 \end{pmatrix}$$

⟨b⟩ Under what conditions on $\underline{b} \in \mathbb{R}^3$ and on $\underline{c} \in \mathbb{R}^4$ will each of the systems $A\underline{x} = \underline{b}$ and $A^T\underline{y} = \underline{c}$ have a solution?

(LU)
$$A \xrightarrow{\begin{pmatrix} 1 & & \\ -1/2 & 1 & \\ 0 & 0 & 1 \end{pmatrix} = L_1} \begin{pmatrix} 2 & 1 & 4 & -2 \\ 0 & 5/2 & -1 & 2 \\ 0 & 5 & -2 & 4 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{pmatrix} = L_2} \begin{pmatrix} 2 & 1 & 4 & -2 \\ 0 & 5/2 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

Now $L = L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & & \\ 1/2 & 1 & \\ 0 & 2 & 1 \end{pmatrix}$

$L^{-1} = L_2 L_1 = \begin{pmatrix} 1 & & \\ -1/2 & 1 & \\ 1 & -2 & 1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 1 & & \\ -1/2 & 1 & \\ 1 & -2 & 1 \end{pmatrix}}_{L^{-1}} \underbrace{\begin{pmatrix} 2 & 1 & 4 & -2 \\ 1 & 3 & 1 & 1 \\ 0 & 5 & -2 & 4 \end{pmatrix}}_{A} = U$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 1 & 4 & -2 \\ 0 & 5/2 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{U}$$

(i) $\mathcal{R}(A) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \cancel{\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}} \right\}$ dim = 2 $\left\{ \begin{array}{l} (x_1, x_2 \text{ basic}) \\ (x_3, x_4 \text{ free}) \end{array} \right.$

(ii) $\mathcal{R}(A^T) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5/2 \\ -1 \\ 2 \end{pmatrix} \right\}$ dim = 2

(iii) $\mathcal{N}(A)$: $\begin{pmatrix} 2 & 1 \\ 0 & 5/2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -x_3 \begin{pmatrix} 4 \\ -1 \end{pmatrix} - x_4 \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ $\left\{ \begin{array}{ll} x_3 = 1, \ x_4 = 0 & x_2 = 2/5, \ x_1 = -11/5 \\ x_3 = 0, \ x_4 = 1 & x_2 = -4/5, \ x_1 = 7/5 \end{array} \right.$

$= \text{span}\left\{ \begin{pmatrix} -11/5 \\ 2/5 \\ 1 \\ 0 \end{pmatrix} = \bar{n}_1, \begin{pmatrix} 7/5 \\ -4/5 \\ 0 \\ 1 \end{pmatrix} = \bar{n}_2 \right\}$ dim = 2

(iv) $\mathcal{N}(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \vec{N}_1 \right\}$

For solvability

$Ax = b$ solvable $\Leftrightarrow b \in \mathcal{R}(A)$
$\Rightarrow b \perp \mathcal{N}(A^T)$

i.e. $\langle b, N_1 \rangle = 0$

$A^T y = c$ solvable $\Leftrightarrow c \in \mathcal{R}(A^T) \Rightarrow c \perp \mathcal{N}(A)$: $\left\{ \begin{array}{l} \langle c, \bar{n}_1 \rangle = 0 \\ \langle c, \bar{n}_2 \rangle = 0 \end{array} \right.$

(e) $I + A$ is nonsingular
(f) $(I+A)^{-1}(I-A)$ is unitary

④ A (real) matrix A is skew symmetric if $a_{ij} = -a_{ji}$, $1 \le i, j \le n$. Show that such a matrix:

(a) Is a normal matrix (i.e. $AA^H = A^H A$).
(b) Has $-\lambda$ as an eigenvalue if λ is an eigenvalue.
(c) Is singular if it is of odd order (=n is odd).
(d) Has pure imaginary eigenvalues.

(a) $A = -A^T \Rightarrow AA^T = -A^2 = A^T A$; normal ($A^T = A^H$ for A real)

(b) Let λ evalue, $\bar{v}$ eigenvector: $A\bar{v} = \lambda \bar{v}$

But then $v^T A^T = \lambda \bar{v}^T \Rightarrow \bar{v}^T A = -\lambda \bar{v}^T$

~~Since~~ i.e. $\bar{v}^T$ is a left eigenvector of A w. evalue $-\lambda$.

(alternatively: $\det A^T = \det A = \det(-A)$

i.e. $|A^T + \lambda I| = |-A + \lambda I| = |-(A - \lambda I)| = (-1)^n |A - \lambda I| = 0$

(c) $\det(A^T) = \det A = \det(-A) = (-1)^n \det A$
general

if n is even, trivial; but if n odd, $\det A = -\det A = 0$.

(d) A real, $A\bar{v} = \lambda \bar{v} \longrightarrow A\bar{v}^* = \lambda^* \bar{v}^*$

λ^* is evalue (true for all real).

i.e. for any evalue, $\lambda \to \lambda^*$

Now $v^* A^H = \lambda^* v^H \Rightarrow v^H A^H v = \lambda^* v^H v$

but $v^H A v = \lambda v^H v$

$$v^H (A^H + A) v = (\lambda^* + \lambda) v^H v = 0 \Rightarrow \boxed{\lambda^* = -\lambda}$$

⑤ Find the pseudoinverse A^+ for

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

and use A^+ to solve $A\underline{x} = \underline{b}$ for

$$\text{(a) } \underline{b} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \; ; \quad \text{(b) } \underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Method 1 (SVD)

$$A^TA = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 5 & -4 \\ 1 & -4 & 5 \end{pmatrix}$$

$$|A^TA - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 5-\lambda & -4 \\ 1 & -4 & 5-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -4 \\ 1 & 5-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 5-\lambda \\ 1 & -4 \end{vmatrix}$$

$$= (2-\lambda)\{(5-\lambda)^2 - 16\} - 2\{(5-\lambda)+4\}$$

$$\lambda^2 - 10\lambda + 9 = (\lambda-1)(\lambda-9)$$

$$= (2-\lambda)(\lambda-1)(\lambda-9) + 2(\lambda-9)$$

$$= (\lambda-9)\{(2-\lambda)(\lambda-1)+2\} = -(\lambda-9)\{\lambda^2 - 3\lambda\} = -(\lambda-9)(\lambda-3)\lambda$$

$$\lambda = 0, 3, 9$$

$$\lambda_1 = 9: \begin{pmatrix} -7 & 1 & 1 \\ 1 & -4 & -4 \\ 1 & -4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \bar{0} \rightarrow \begin{pmatrix} -7 & 1 & 1 \\ 0 & * & * \\ 0 & \cancel{*} & \cancel{*} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \begin{cases} c = 1 \\ b = -1 \\ a = 0 \end{cases}$$

$$\lambda_1 = 9, \bar{V}_1 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\lambda_2 = 3: \begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & -4 \\ 1 & -4 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \bar{0} \Rightarrow \begin{matrix} c = 1 \\ b = 1 \\ a = 2 \end{matrix}$$

$$\lambda_2 = 3, \; \bar{V}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = 0: \begin{pmatrix} 2 & 1 & 1 \\ 1 & 5 & -4 \\ 1 & -4 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 1 \\ 0 & 9/2 & -9/2 \\ 0 & -9/2 & 9/2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$c = 1, \; b = 1; \; a = -1: \quad \bar{V}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ \end{pmatrix}$$

$\sigma_1 = 3$
$\sigma_2 = \sqrt{3}$

i.e. $\vec{V} = \begin{pmatrix} 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$; $\Sigma = \begin{pmatrix} 3 & & \\ & \sqrt{3} & \\ & & 0 \end{pmatrix}$

$\bar{u}_i = \frac{1}{\sigma_i} A v_i$:

$$\bar{u}_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix}$$

$$\bar{u}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

For $\bar{u}_3$: $A^T u_3 = 0$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \bar{0} \Rightarrow \begin{matrix} c = 1 \\ b = 2 \\ a = -2 \end{matrix}$$

$$\bar{u}_3 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$\begin{matrix} & U & \Sigma & V^H \end{matrix}$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{pmatrix} = \underset{U'}{\begin{pmatrix} \boxed{\begin{matrix} -1/3\sqrt{2} & 1/\sqrt{2} \\ 1/3\sqrt{2} & 1/\sqrt{2} \\ -4/3\sqrt{2} & 0 \end{matrix}} & \begin{matrix} 1/3 \\ 2/3 \\ -2/3 \end{matrix} \end{pmatrix}} \begin{pmatrix} 3 & & \\ & \sqrt{3} & \\ & & 0 \end{pmatrix} \overset{V'}{\begin{pmatrix} \boxed{\begin{matrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{matrix}} \\ \begin{matrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{matrix} \end{pmatrix}}$$

$$A^+ = V \Sigma^+ U^H \; ; \; \Sigma^+ = \begin{pmatrix} 1/3 & & \\ & 1/\sqrt{3} & \\ & & 0 \end{pmatrix}$$

$$A^+ \begin{pmatrix} 3 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}\begin{pmatrix} 1/3 & 0 \\ 0 & 1/\sqrt{3} \end{pmatrix}\underbrace{\begin{pmatrix} -1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}}_{\begin{pmatrix} -3\sqrt{2} & -2\sqrt{2}/3 \\ 2\sqrt{2} & \sqrt{2} \end{pmatrix}}$$

$$\begin{pmatrix} -\sqrt{2} & -2\sqrt{2}/9 \\ 2\sqrt{2}/\sqrt{3} & \sqrt{2}/\sqrt{3} \end{pmatrix}$$

$$(\bar{x}_1 \bar{x}_2) = \begin{pmatrix} 4/3 & 2/3 \\ 5/3 & 5/9 \\ -1/3 & 1/9 \end{pmatrix}$$

Method II $\quad A = BC \qquad$ rank $= 2$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow[L_1]{\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow[L_2]{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 2 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = U \Rightarrow C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$L = L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & -2 & 1 \end{pmatrix} \Rightarrow B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -2 \end{pmatrix}$$

$$CC^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = BC$$

$$(CC^T)^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$B^T B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

$$A^+ = C^T (CC^T)^{-1} (B^T B)^{-1} B^T$$

$$(B^T B)^{-1} = \frac{1}{9} \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \frac{1}{9} \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\frac{1}{27} \begin{pmatrix} 9 & 0 \\ 3 & 3 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \cancel{\tfrac{1}{27}} \; \frac{1}{9} \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & -2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$$

then

$$(\bar{x}_1, \bar{x}_2) = A^+ \begin{pmatrix} 3 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 12 & 6 \\ 15 & 5 \\ -3 & 1 \end{pmatrix}$$

⑥ Let $\underline{u} \in \mathbb{C}^n$ be a unit vector, i.e. $\|u\|_2 = 1$.
Show that the $n \times n$ matrix

$$H = I - 2uu^H$$

(a) Is Hermitian
(b) Is unitary
(c) Find the eigenvalues of H and their multiplicities.
(d) If $\underline{v}, \underline{w} \in \mathbb{R}^n$ are real and satisfy

$\|v\|_2 = \|w\|_2$, find a (real) unit vector $\underline{u}$

such that $\underline{w} = H\underline{v}$ provided that $\underline{w} \neq \underline{v}$
(and $n > 1$).

(a) $H^H = (I - 2uu^H)^H = I^H - 2(uu^H)^H = I - 2uu^H = H$

(b) $HH^H = HH = (I - 2uu^H)(I - 2uu^H) =$
$I - 4uu^H + 4u(\underbrace{u^H u}_{=1})u^H = I$

(c) $(I - 2uu^H)v = \lambda v$
$\Rightarrow$ (i) $v \in \{u\}^\perp$: $(I - 2uu^H)v = Iv = v$
$\lambda = 1$, multiplicity $(n-1)$
(ii) $v = u$: $(I - 2u(u^H)u) = u - 2u = -u$
$\lambda = -1$, multiplicity 1

(d) $Hv = w \Rightarrow v - 2u\langle u, v\rangle = w \Rightarrow 2u\langle u, v\rangle = v - w$
Since $\|u\|_2 = 1 \Rightarrow \boxed{u = \pm \dfrac{v - w}{\|v - w\|}}$