Math. 464
Solutions, Set 11

5.6 (4, 7, 13, 24, 28, 29)
Review, 5 (8, 11, 20) ~~(10.20, 10.21)~~

p.315, 5.6.4

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & \ddots & 1 \\ 0 & 1 & 2 \end{pmatrix} \; ; \; B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Let $M = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -(-1)^n \end{pmatrix} = M^{-1}$

$$MA = \begin{pmatrix} 2 & 1 & & \\ -1 & -2 & -1 & \\ & 1 & 2 & 1 \\ & & & \ddots \end{pmatrix} \; ; MAM^{-1} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} = B$$

i.e. $MAM^{-1} = B$ and $A \sim B$.

5.6.7 (A Givens rotation)

$$M = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \; ; \; M^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}$$

~~What to rotate~~ : note that M^{-1} leaves 3rd row unchanged in multiplication $M^{-1}A$. So, it is enough ~~tho~~ to get things to work in multiplication AM:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3.1) entry: $g\cos\theta + h\sin\theta = 0 \Rightarrow \boxed{\tan\theta = -g/h}$

$$\theta = \tan^{-1}(-g/h)$$

5.6.13 $\frac{d}{dx}\left(a+bx+cx^2\right) = b + 2cx + 0x^2$

(a) $D\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix}$: $D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $D^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $D^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\frac{d^2}{dx^2}\left(a+bx+cx^2\right) = 2c + 0x + 0x^2$; $\frac{d^3}{dx^3}\left(a+bx+cx^2\right) = 0 + 0x + 0x^2$

(In the vector space of 2nd degree polynomials, D^3 is identically zero since the third derivative of any quadratic vanishes identically.

(c) $\left|D - \lambda I\right| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0$: $\lambda = 0$ is only evalue.

eigenvectors: $(D - \underset{0}{\underset{\shortparallel}{\lambda}} I)v = 0 \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0} \Rightarrow$ $v_2 = 0$, $2v_3 = 0$, v_1 arbitrary

only eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, evalue 0.

Generalized evectors:

$$\underset{\vec{v}_2}{D\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} = \underset{\vec{v}_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \; ; \quad \underset{\vec{v}_3}{D\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}} = \underset{\vec{v}_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

i.e. we have a single chain: $D\vec{v}_3 = \vec{v}_2$, $D\vec{v}_2 = \vec{v}_1$, $D\vec{v}_1 = 0$

i.e. $D\left(\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3\right) = \left(0 \; v_1 \; v_2\right) = \left(\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3\right)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

5.6.24 $(T-\lambda_1 I)(T-\lambda_2 I)(T-\lambda_3 I)$

($\boxed{5.6.25}$ is trivial!)

$$= \underbrace{\begin{pmatrix} 0 & t_{12} & t_{13} \\ 0 & \lambda_2-\lambda_1 & t_{23} \\ 0 & & \lambda_3-\lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1-\lambda_2 & t_{12} & t_{13} \\ 0 & 0 & t_{23} \\ 0 & & \lambda_3-\lambda_2 \end{pmatrix}} \begin{pmatrix} \lambda_1-\lambda_3 & t_{12} & t_{13} \\ 0 & \lambda_2-\lambda_3 & t_{23} \\ 0 & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & t_{12}t_{23}+t_{13}(\lambda_3-\lambda_2) \\ 0 & 0 & t_{23}(\lambda_3-\lambda_2)+t_{23}(\lambda_2-\lambda_1) \\ 0 & 0 & (\lambda_3-\lambda_1)(\lambda_3-\lambda_2) \end{pmatrix} \cdot$$

$$= 0$$

Now, any matrix is unitarily similar to triangular:

$$A = UTU^{-1} \iff T = U^{-1}AU$$

Then, if $(T-\lambda_1 I)\cdots(T-\lambda_n I) = 0 \Rightarrow$

$$\Rightarrow U\left[(T-\lambda_1 I)\cdots(T-\lambda_n I)\right]U^{-1} = 0$$

$$\Rightarrow (A-\lambda_1 I)\cdots(A-\lambda_n I) = 0$$

i.e. if $\det(A-\lambda I) = 0 \equiv P(\lambda) = 0$

then $\boxed{P(A) = 0}$

(Note: above we "proved" it only for 3×3; but the proof easily generalizes: show, if $T = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \ \lambda_n \end{pmatrix}$

$(T-\lambda_1 I)(T-\lambda_2 I)$ has first two columns $= 0$

Then $(T-\lambda_1 I)(T-\lambda_2 I)(T-\lambda_3 I)$ has first 3 cols $= 0$,

$(T-\lambda_1 I)\cdots(T-\lambda_k I)$ has first k-cols. $= 0$ etc.

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5.6.13 $\frac{d}{dx}\left(a+bx+cx^2\right) = b+2cx+0x^2$

(a) $D\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix}$: $D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $D^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $D^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\frac{d^2}{dx^2}\left(a+bx+cx^2\right) = 2c+0x+0x^2$; $\frac{d^3}{dx^3}\left(a+bx+cx^2\right) = 0+0x+0x^2$

(In the vector space of 2nd degree polynomials, D^3 is identically zero since the third derivative of any quadratic vanishes identically.

(c) $|D-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0$: $\lambda = 0$ is only evalue.

eigenvectors: $(D-\underset{\overset{\shortparallel}{0}}{\lambda} I)v = 0 \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0} \Rightarrow$ $v_2 = 0$, $2v_3 = 0$, v_1 arbitrary

only eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, evalue 0.

Generalized evectors:

$$D\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_2} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{v}_1} ; \quad D\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}}_{\vec{v}_3} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_2}$$

i.e. we have a single chain: $D\vec{v}_3 = \vec{v}_2$, $D\vec{v}_2 = \vec{v}_1$, $D\vec{v}_1 = 0$

i.e. $D\left(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\right) = \left(0\ v_1\ v_2\right) = \left(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\right)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

5.6.28 Solve $u' = Ju = \begin{pmatrix}5 & 1\\ 0 & 5\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}$, $\vec{u}(0) = \begin{pmatrix}1\\ 2\end{pmatrix}$
by back sub:

$$u_1' = 5u_1 + u_2 \Rightarrow u_1' - 5u_1 = u_2 = 2e^{5t}$$
$$u_2' = 5u_2 \Rightarrow u_2 = e^{5t}u_2(0) = 2e^{5t} \uparrow$$
$$\Rightarrow \left(e^{-5t}u_1\right)' = 2 \Rightarrow \underbrace{e^{-5t}u_1 = 2t + C}\Rightarrow u_1(t) = 2te^{5t} + Ce^{5t}$$
$$u_1(0) = 1 = C \quad : \quad \vec{u} = \begin{pmatrix}1\\ 2\end{pmatrix}e^{5t} + \begin{pmatrix}2\\ 0\end{pmatrix}te^{5t}$$

5.6.29 Find A^{10}, e^A; $A = \begin{pmatrix}14 & 9\\ -16 & -10\end{pmatrix} = \begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix} = MJM^{-1}$

A (and J) satisfy the minimal polynomial $P(\lambda) = (\lambda - 2)^2$:

$$P(J) = J^2 - 4J + 4 = 0 \Rightarrow J^2 = 4(J-1).$$

The easiest way to compute the k-th power of A is through the formula:

$$J^k = \begin{pmatrix}\lambda^k & k\lambda^{k-1} & \cdots \\ & \ddots & k\lambda^{k-1}\\ 0 & & \lambda^k\end{pmatrix}$$

So: $\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}^{10} = \begin{pmatrix}2^{10} & 10\cdot 2^9\\ 0 & 2^{10}\end{pmatrix}$; $A^{10} = \cancel{M}\, 2^{10} M\begin{pmatrix}2 & 5\\ 0 & 2\end{pmatrix}M^{-1}$

$$= 2^{10}\begin{pmatrix}6 & 45\\ 80 & 58\end{pmatrix}$$

$$e^A = \cancel{I} + Me^JM^{-1} = e^2M\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}M^{-1} = e^2\begin{pmatrix}13 & 9\\ -16 & -11\end{pmatrix}$$

$$e^{Jt} = I + Jt + \frac{1}{2}J^2t^2 + \cdots + \frac{1}{k!}J^k + \cdots = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + t\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} + \cdots + \frac{t^k}{k!}\begin{pmatrix}\lambda^k & k\lambda^{k-1}\\ & \lambda^k\end{pmatrix} + \cdots$$

$$= \begin{pmatrix}\underbrace{1 + \lambda t + \frac{1}{2}\lambda^2t^2 \cdots + \frac{t^k}{k!}\lambda^k + \cdots}_{e^{\lambda t}} & \underbrace{0 + t + \lambda t^2 \cdots + \lambda^k t^{k+1}}_{te^{\lambda t}}\\ 0 & e^{\lambda t}\end{pmatrix} = \begin{pmatrix}e^{\lambda t} & te^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix}$$

$$t=1: \; = \begin{pmatrix}e^{\lambda} & e^{\lambda}\\ 0 & e^{\lambda}\end{pmatrix} = e^{\lambda}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix};$$

5.6.39 Prove that A^T is similar to A:

(i) Let $J^{n\times n} = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix}$, $J^T = \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & 1 & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix}$

Note that if we reverse the rows of J we arrive at the same result as when we reverse the columns of J^T:

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 1 \\ & 0 & & & \lambda \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & & \lambda \\ & \diagup & 1 \\ \lambda & 1 & 0 \end{pmatrix} = J \begin{pmatrix} 0 & & 1 \\ & \diagup & \\ 1 & & 0 \end{pmatrix} = JP$$

$$J^T = \begin{pmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & & \lambda \\ & \diagup & 1 \\ \lambda & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & 1 \\ & \diagup & \\ 1 & & 0 \end{pmatrix} J^T = PJ^T$$

$$P = P^{-1}$$

i.e. $JP = PJ^T$ or $J = PJ^TP^{-1} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} J^T \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $P^{-1} = P$

(ii) For a general $J^{n\times n}$; each block need its own reversal:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix} = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_k \end{pmatrix} \begin{pmatrix} J_1^T & & \\ & \ddots & \\ & & J_k^T \end{pmatrix} \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_k \end{pmatrix}$$

where J_i is $n_i \times n_i$, $P_i^{n_i\times n_i} = \begin{pmatrix} 0 & & 1 \\ & \diagup & \\ 1 & & 0 \end{pmatrix}$ with $\sum_{i=1}^{k} n_i = n$

(iii) For general A:

$$A = MJM^{-1} = M\left(PJ^TP^{-1}\right)M^{-1} = (MP)J^T(MP)^{-1}$$

while $A^T = (M^{-1})^T J^T M^T = (M^T)^{-1}\left\{(MP)^{-1} A (MP)\right\} M^T$

$$\Rightarrow \boxed{A^T = (MPM^T)^{-1} A (MPM^T)}$$

Review 5.8

(a) False: row operations (such as a row exchange) are achieved by left-multiplication by permutation matrix. Right mult. will affect columns.

Counter example: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $|A-\lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d) + (ad-bc) = 0$

while $B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$; $|B-\lambda I| = \begin{vmatrix} b-\lambda & d \\ a & b-\lambda \end{vmatrix} = \lambda^2 - (b+c) - (ad-bc) = 0$

These will have different eigenvalues in general.

(b) False; a general triangular matrix with distinct eigenvalues is diagonalizable, i.e. similar to diagonal matrix without itself being diagonal

(c) Any two of: $\{A = A^H\}$, $\{AA^H = I\}$, $\{A^2 = I\}$ imply the third

True: $(A = A^H,\ AA^H = I) \Rightarrow A^2 = I$

$(A = A^H,\ A^2 = I) \Rightarrow AA^H = A^2 = I$

$(AA^H = I,\ A^2 = I) \Rightarrow A = A^{-1} = A^H$

(d) If A, B are diagonalizable, so is AB

False: Consider Counterexample $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\left(\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \right)$

$B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\left(\begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (\lambda - 1)\lambda - 1 = \lambda^2 - \lambda - 1 \right)$

$\lambda = \frac{1 \pm \sqrt{5}}{2}$

Both have distinct evalues $\Rightarrow$ diagonalizable. But

$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which is in Jordan normal form and cannot be diagonalized.

Review 5.11 P projects $\mathbb{R}^n$ to $S \subset \mathbb{R}^n$.

(1) Every vector in S is an eigenvector
(2) " " " $S^\perp$ " " "

Indeed, let $x \in \mathbb{R}^n \Rightarrow x = y + z$, $y \in S$, $z \in S^\perp$

$$Px = P(y+z) = \underset{\overset{\shortparallel}{y}}{Py} + \underset{\overset{\shortparallel}{0}}{Pz} = y$$

In general, $y \in S \Rightarrow Py = y$ (evector, evalue 1)

$z \in S^\perp \Rightarrow Pz = 0$ (" , evalue 0)

Eigenvalues are: 1 (geometric multiplicity : $\dim S$)

0 (" : $\dim S^\perp$)

Review 5.20 If $K^H = -K$ (skew-Hermitian); orthogonal evectors, imaginary evalues

(a) $K - I$ has eigenvalues $\lambda - 1$, where λ is an evalue of K.
since λ is imaginary, $K - I$ has no zero eigenvalues
$\Rightarrow K - I$ invertible

(b) $KK^H = -K^2 = K^H K$ i.e. K is normal. By Schur, $\exists\, U$ unitary such that $K = UTU^H$; T is triangular + normal
$\Rightarrow T$ is diagonal
$T = \Lambda$, Λ has imaginary diagonal elements

(c) $e^{-\Lambda t} = e^{\Lambda^H t}$; $e^{-\Lambda t} e^{\Lambda t} = I = (e^{\Lambda t})^H (e^{\Lambda t}) = e^{-\Lambda t} e^{\Lambda t}$.

(d) $(e^{Kt})^H = e^{K^H t} = e^{-Kt}$ so $(e^{Kt})^H (e^{Kt}) = e^{-Kt} e^{Kt} = I$

$\Rightarrow e^{Kt}$ unitary.