Set 9 $\left\{ \left[(4.4.5),\ 5.1(5),\ 5.2(4,6,8,13),\ 5.4(1,4,5) \ \text{+ 2 supplementary problems} \right] \right.$ S/EX/8

4.4.5 $A = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$; $A\vec{x} = \lambda \vec{x} \Rightarrow \lambda = 3, 1, 0$

$\vec{e}_3 : \begin{pmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \vec{e}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $\vec{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$B = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$; $\det\begin{pmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{pmatrix} = 0 \Rightarrow (2-\lambda)\lambda^2 - 4(2-\lambda) = 0$

$\Rightarrow (2-\lambda)(\lambda-2)(\lambda+2) = 0 \Rightarrow \lambda = -2$, $\lambda = 2$ (order 2)

evectors: $(B+2I)\vec{e}_{-2} = 0 \Rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow$ $b = 0$, $a + c = 0 \Rightarrow a = 1, c = -1$

b = arbitrary

$(B-2I)\vec{e}_2 = 0 \Rightarrow \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow$ $a = c = 1$

i.e. $\vec{e}_{-2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\vec{e}_{2,1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{e}_{2,2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (i.e. there are two independent evectors of $\lambda = 2$)

$A: \lambda_1\lambda_2\lambda_3 = 0 = a_{11}a_{22}a_{33}$; $\lambda_1+\lambda_2+\lambda_3 = 4 = \text{Tr} A$ | $B: \lambda_1\lambda_2\lambda_3 = 8 = \det B$, $\lambda_1+\lambda_2+\lambda_3 = 2 = \text{Tr} B$.

5.1.11 Since $\det(A^T - \lambda I) = \det(A-\lambda I)^T = \det(A-\lambda I)$, ...

5.1.18

$Av_0 = 0$, $Av_1 = v_1$, $Av_2 = 2v_2$; Then v_0 spans $\mathcal{N}(A)$

and if $Ax = v_0$ had a solution, $x = av_0 + bv_1 + cv_2$ then

$Ax = a \cdot 0 + bv_1 + 2cv_2 = v_0$, impossible as v_0, v_1, v_2 indep.

while $x = v_1 + \frac{1}{2}v_2$ solves $Ax = v_1 + v_2$.

5.2.4 if $A = \begin{pmatrix} 1 & x & y \\ 0 & 2 & z \\ 0 & 0 & 7 \end{pmatrix}$ then $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

and diagonalization is assured since each eigenvalue has a corresponding e-vector, independent of the others, etc.

5.2.6 (a) $A^2 = I$; if λ is evalue $\Rightarrow \lambda^2 = 1$, i.e. $\lambda = \pm 1$

(b) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow \det A = a_{11}a_{22} - a_{12}a_{22}$, $\mathrm{Tr}A = a_{11} + a_{22}$

Now $A^2 = I \to \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{21}a_{22} & a_{21}a_{12} + a_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

~~with~~ now let $\det A = \delta$, $\mathrm{Tr}A = \tau$; so

$$a_{12}a_{21} + a_{11}^2 = -\delta + \tau(a_{11}$$
$$a_{21}a_{12} + a_{22}^2 = -\delta + \tau a_{22}$$
$$a_{11}a_{12} + a_{12}a_{22} = a_{12}\tau$$
$$a_{21}a_{11} + a_{21}a_{22} = a_{21}\tau$$

$$A^2 = \begin{pmatrix} -\delta + \tau a_{11} & a_{12}\tau \\ a_{21}\tau & -\delta + \tau a_{22} \end{pmatrix}$$

$= -\delta I + \tau A = I$ or $\boxed{\tau A = (1+\delta) I}$

So (i) $\tau = 0 \Leftrightarrow \delta = -1$, i.e. $\lambda_1 = 1$, $\lambda_2 = -1$.

(ii) if $\delta \neq -1 \Rightarrow \delta = 1$, then $\tau A = 2I$;

but then $a_{12} = a_{21} = 0$ and $\tau = \begin{cases} 2, & \Rightarrow A = I \\ -2 & \Rightarrow A = -I \end{cases}$

which are excluded

$\therefore \Rightarrow \underline{\tau = 0 \text{ and } \delta = -1}$

5.2.18 $A = uv^T$

(a) $Au = (uv^T)u = u(v^Tu) = (v^Tu)u$; $\lambda = v^Tu$

(b) Since A is rank 1 (obviously: all its rows are multiples of v^T), its null space has dimension $n-1$. If $\lambda \neq 0$ (i.e. $v^Tu \neq 0$), then $\{u\}^\perp$ gives $n-1$ evectors of eigenvalue 0. If $v^Tu = 0$, then u is contained in the null space, ~~Thus~~ and $\lambda = 0$ has multiplicity n. Now there is a vector w s.t. $(uv^T)w \neq 0$.

In fact $(uv^T)w = v$

5.2.8

$A = u v^T$ (matrix); (a) $Au = u(v^T u) = (v^T u) u$ ($= \lambda$, number)

i.e. u is evector with evalue λ.

(b) Now, if $x \in \{v\}^\perp \Rightarrow Ax = 0$ and, as expected, $\mathcal{N}(A)$ has dimension $n-1$ (since rows of A are multiples of v^T, it follows $\operatorname{rank}(A) = 1$). Thus we have ~~tha~~ at least $n-1$ eigenvectors of the eigenvalue 0. If $v^T u \neq 0$, then u is an eigenvector of evalue λ and there are exactly $n-1$ evectors of evalue 0.

If $v^T u = 0$, then $\lambda = 0$ is an eigenvalue of (algebraic) multiplicity n. However there are only $n-1$ evectors of evalue 0 (since $Av = u(v^T v) = \|v\|^2 u \neq 0$, so that there is always a direction not annihilated by A, unless $v \equiv 0$).

(c) $\operatorname{Tr} A = \sum \lambda = (v^T u)$; or

$\operatorname{Tr} A = \sum_{i=1}^{n} u_i v_i = v^T u$

5.2.10

A (a 3×3 matrix) has evalues 1, 2, 4.

Then A^2 has evalues $1^2 = 1$, $2^2 = 4$, $4^2 = 16$ and $\operatorname{Tr} A^2 = 1 + 4 + 16 = 21$. Also $(A^{-1})^T$ has evalues $1, \frac{1}{2}, \frac{1}{4}$, so $\det (A^{-1})^T = \frac{1}{8}$.

5.2.13 $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$; $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = 0$

$\Rightarrow \lambda = 5 \pm 4 < {9 \atop 1}$

$\vec{e}_9$: $\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = b = 1$; $\vec{e}_9 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{e}_1$: $\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = -b = 1$; $\vec{e}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$A\,S = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ 9 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow S^{-1} A S = \Lambda = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^{1/2} = R \text{ ; } \quad S^{-1} R^2 S = (S^{-1} R S)(S^{-1} R S) = M^2$$

with $M^2 = \Lambda$; Then $S^{-1} R S = M$

So, since $M = \begin{pmatrix} \pm 3 & 0 \\ 0 & \pm 1 \end{pmatrix}$

($\pm$ unrelated, 4 cases)

$$R = S M S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \pm 3 & 0 \\ 0 & \pm 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \pm 3 & \pm 3 \\ \pm 1 & \mp 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \pm 3 \pm 1 & \pm 3 \mp 1 \\ \pm 3 \mp 1 & \pm 3 \pm 1 \end{pmatrix}$$

$R_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $R_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$,

(top, top) signs; (top, bottom)

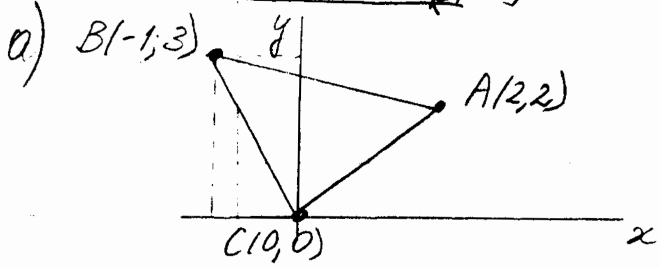
$R_3 = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$, $R_4 = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

(bottom, top); (bottom, bottom)

HW #

Ex. 4.4 (8/5)

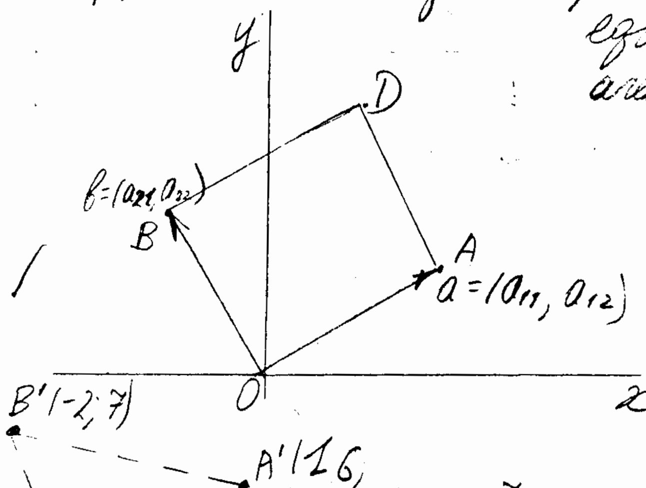
a)



By regarding it as half of a parallelogram, explain why

$$\text{area}(ABC) = \frac{1}{2}\det\begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix}$$

The area of a parallelogram ABCD is equal to

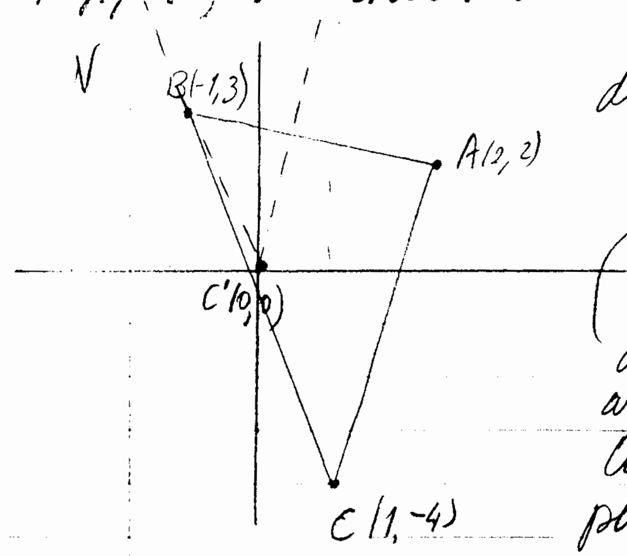


$$\text{area}(ABCD) = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \det\begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix}$$

From geometry:

$$\text{area}(\triangle ABC) = \frac{1}{2}\text{area}(ABCD) = \frac{1}{2}\det\begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix}$$

b) Suppose, the third vertex is $C=(1,-4)$ instead of $C(0,0)$



$$\det\begin{pmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 6 & 0 \\ -2 & 7 & 0 \\ 1 & -4 & 1 \end{pmatrix} =$$

$$= 1\cdot(-1)^3\det\begin{pmatrix} 1 & 6 \\ -2 & 7 \end{pmatrix} = \det\begin{pmatrix} 1 & 6 \\ -2 & 7 \end{pmatrix}$$

det does not change (rule 5)

$\text{area}(A'B'C'D') = \det\begin{pmatrix} 1 & 6 \\ -2 & 7 \end{pmatrix}$

All vertices of $\triangle A'B'C'$ are parallel to the same vertices of $\triangle ABC$, and the length is the same

$S_{\triangle ABC} = S_{\triangle A'B'C'}$ ($\triangle A'B'C'$ is just moved to the origin)

$$\Rightarrow S(\triangle ABC) = \frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \frac{1}{2}\det\begin{pmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{pmatrix} =$$

$$= \frac{1}{2}\det\begin{pmatrix} 1 & 6 \\ -2 & 7 \end{pmatrix}$$

P.286, 5.4.1 > Find evals/vecs. for e^{At}

if $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$\det(A-\lambda I) = \det\begin{pmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = (\lambda+1)^2 - 1 = 0 \Rightarrow \lambda = -1 \pm 1 \begin{cases} 0 \\ -2 \end{cases}$

$e_\lambda : (-1-\lambda \quad 1)\begin{pmatrix} a \\ b \end{pmatrix} = 0 \;:\; e_\lambda = \begin{pmatrix} 1 \\ \lambda+1 \end{pmatrix}$

$e_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} ; \; e_{-2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$:

e^{At} has same eigenvectors & eigenvalues $\begin{cases} e^{0t} = 1 \\ e^{-2t} \end{cases}$

5.4.4 > P projection: $P^n = P$;

$e^P = I + P + \frac{1}{2}P^2 + \dots + \frac{1}{n!}P^n = I + \left(1 + \frac{1}{2} + \dots + \frac{1}{n!} + \right)P$

$= I + (e^1 - 1)P \simeq I + 1.718\,P$

5.4.5 > (a) $e^{A(t+T)} = S e^{\Lambda(t+T)} S^{-1} = (S e^{\Lambda t} S^{-1})(S e^{\Lambda T} S^{-1})$

$= e^{At} e^{AT}$

(b) $e^A = I + A + \frac{1}{2}A^2 + \dots = I + A$ (since $A^2 = \dots = A^k = 0$)

$e^B = I + B + \frac{1}{2}B^2 + \dots = I + B$ (similarly, $B^2 = 0$ etc)

So $e^A e^B = (I+A)(I+B) = I + A + B + AB$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

while $e^B e^A = (I+B)(I+A) = I + B + A + BA$ #

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

However: $e^{A+B} = e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = I + C + \frac{1}{2}C^2 + \dots + \frac{1}{n!}C^n + \dots$

let $A+B = C$

$\left(C^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \; ; \; C^3 = -C, \; C^4 = I \right)$

$= I\left(1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} \dots\right) +$

$+ C\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \dots\right)$

$= I \cos 1 + C \sin 1 \neq e^A e^B$

i.e. $e^{A+B} = \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix} \neq e^A e^B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$= e^B e^A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

5.4.14 $\dfrac{d^2 u}{dt^2} = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix} u$

let $\underline{u} = \begin{pmatrix} a \\ b \end{pmatrix} e^{i\omega t}$; $-\omega^2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -5+\omega^2 & 4 \\ 4 & -5+\omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $(\omega^2-5)^2 - 16 = 0$

$\omega^2 = 5 \pm 4 = \begin{cases} 9 \\ 1 \end{cases}$

$\omega_1 = \pm 3$; $\omega_2 = \pm 1$

w8/1/17 Then: $\omega_1 = \pm 3$; $\begin{pmatrix} -5+9 & 4 \\ 4 & -5+9 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{e}_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\omega_1 = \pm 1$ $\begin{pmatrix} -5+1 & 4 \\ 4 & -5+1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So: $\boxed{\underline{u}(t) = (a_1 \cos 3t + b_1 \sin 3t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (a_2 \cos t + b_2 \sin t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Supplementary problems

(1) Show that (5) $\Rightarrow$ (1) in notes on ~~Cayley~~ Geometry

$(n+2)\times(n+2)$

$$\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & -\frac{1}{2}d_{01}^2 \cdots & -\frac{1}{2}d_{0n}^2 \\ \vdots & -\frac{1}{2}d_{01}^2 & 0 \cdots & -\frac{1}{2}d_{1n}^2 \\ 1 & -\frac{1}{2}d_{0n}^2 & \cdots & 0 \end{vmatrix} = \left(-\frac{1}{2}\right)^{n+1} \begin{vmatrix} 0 & 1 & \cdots & 1 \\ -2 & 0 & d_{01}^2 \cdots & d_{0n}^2 \\ -2 & d_{01}^2 & 0 \cdots & d_{1n}^2 \\ \vdots & & & \\ -2 & d_{0n}^2 & \cdots & 0 \end{vmatrix}$$

(multiply each row by -2 except for 1st row)

multiply 1st column by $\left(-\frac{1}{2}\right)$

$$= \left(-\frac{1}{2}\right)^{n} \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & d_{01}^2 \cdots & d_{0n}^2 \\ \vdots & d_{01}^2 & 0 \cdots & d_{1n}^2 \\ 1 & d_{0n}^2 & \cdots & 0 \end{vmatrix}$$

So

$$V_n^2 = -\frac{1}{(n!)^2} \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & -\frac{1}{2}d_{01}^2 \cdots & -\frac{1}{2}d_{0n}^2 \\ \vdots & & & \\ 1 & & & \end{vmatrix} = \frac{(-1)^{n+1}}{2^n (n!)^2} \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & d_{01}^2 \cdots & d_{0n}^2 \\ \vdots & d_{01}^2 & 0 \cdots & d_{1n}^2 \\ 1 & d_{0n}^2 & d_{1n}^2 \cdots & 0 \end{vmatrix} \quad (1)$$

(5)

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(2) Triangle:

$$V_2^2 = \frac{(-1)^3}{2^2(2!)^2} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 \end{vmatrix} = -\frac{1}{16}\left\{ -\begin{vmatrix} 1 & d_{12}^2 & d_{13}^2 \\ 1 & 0 & d_{23}^2 \\ 1 & d_{23}^2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & d_{13}^2 \\ 1 & d_{12}^2 & d_{23}^2 \\ 1 & d_{13}^2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & d_{12}^2 \\ 1 & d_{12}^2 & 0 \\ 1 & d_{13}^2 & d_{23}^2 \end{vmatrix} \right.$$

expand by 1st row

$= -1/16 \qquad (d_{12} = c,\ d_{13} = b,\ d_{23} = a)$

$$= \frac{1}{16}\left\{ \left(-d_{23}^4 + d_{13}^2 d_{23}^2 + d_{12}^2 d_{23}^2\right) + \left(-d_{13}^2 d_{23}^2 + d_{13}^4 - d_{12}^2 d_{13}^2\right) + \left(d_{12}^2 d_{23}^2 + d_{12}^2 d_{13}^2 - d_{12}^4\right) \right\}$$

$$= \frac{1}{16}\left\{ -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \right\}$$

see ~~back~~ next page

$$(a+b+c)(a+b-c)(a-b+c)(-a+b+c) = -[(a+b)^2 - c^2][(a-b)^2 - c^2] = -[a^2+b^2-c^2+2ab][a^2+b^2-c^2-2ab]$$

$$= -[(a^2+b^2-c^2)^2 - 4a^2b^2] = \ldots$$

(2) Show

$$-V^2 = +\frac{1}{16}\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} = -\frac{1}{16}(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$$

$\parallel$ (from previous page)

$$+\left\{a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2\right\} =$$

$$=\left(a^4+b^4+2a^2b^2\right)+c^4-2b^2c^2-2a^2c^2-4a^2b^2$$

$$=\underbrace{(a^2+b^2)^2+c^4-2c^2(a^2+b^2)}-4a^2b^2$$

$$=(a^2+b^2-c^2)^2-4a^2b^2=\left[a^2+b^2-c^2+2ab\right]\left[a^2+b^2-c^2-2ab\right]$$

$$=\left[(a+b)^2-c^2\right]\left[(a-b)^2-c^2\right]=(a+b+c)(a+b-c)(a-b+c)(a-b-c)$$

$$\therefore \boxed{V^2=\frac{1}{16}(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$$