P.286, 5.4.1 > Find evals/vecs. for e^{At} if $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = (\lambda+1)^2 - 1 = 0 \Rightarrow \lambda = -1 \pm 1 \begin{cases} 0 \\ -2 \end{cases}$$

$$\underline{e}_\lambda : \begin{pmatrix} -1-\lambda & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \; : \; \underline{e}_\lambda = \begin{pmatrix} 1 \\ \lambda+1 \end{pmatrix}$$

$$\underline{e}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \; ; \; \underline{e}_{-2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \; :$$

e^{At} has same eigenvectors & eigenvalues $\begin{cases} e^{0t} = 1 \\ e^{-2t} \end{cases}$

5.4.4 > P projection: $P^n = P$;

$$e^P = I + P + \tfrac{1}{2}P^2 + \dots + \tfrac{1}{n!}P^n = I + \left(1 + \tfrac{1}{2} + \dots + \tfrac{1}{n!} \dots\right)P$$
$$= I + (e^1 - 1)P \simeq I + 1.718\,P$$

5.4.5 > (a) $e^{A(t+T)} = S e^{\Lambda(t+T)} S^{-1} = (S e^{\Lambda t} S^{-1})(S e^{\Lambda T} S^{-1})$
$= e^{At} e^{AT}$

(b) $e^A = I + A + \tfrac{1}{2}A^2 + \dots = I + A$ (since $A^2 = \dots = A^k = 0$)
$e^B = I + B + \tfrac{1}{2}B^2 + \dots = I + B$ (similarly, $B^2 = 0$ etc)

So $e^A e^B = (I+A)(I+B) = I + A + B + AB$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

while $e^B e^A = (I+B)(I+A) = I + B + A + BA$ #
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

However: $e^{A+B} = e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = I + C + \tfrac{1}{2}C^2 + \dots + \tfrac{1}{n!}C^n + \dots$

let $A+B = C$

$\left(C^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \; ; \; C^3 = -C, \; C^4 = I \right)$
$$= I\left(1 - \tfrac{1}{2} + \tfrac{1}{4!} - \tfrac{1}{6!} \dots\right) + C\left(1 - \tfrac{1}{3!} + \tfrac{1}{5!} - \tfrac{1}{7!} \dots\right)$$
$$= I\cos 1 + C \sin 1 \neq e^A e^B$$

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