1 p.427, App. B. 1 Find Jordan forms:

(a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Column space: span $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\longrightarrow$ no intersection:

Null space span $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

(i) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ : eigenvector, evalue 1.

(ii) No intersection, nothing to do $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(iii) Null space outside of column space: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

i.e. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \cancel{\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{Jordan form}} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

(b) $B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ : $C(B) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ : $B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$ :

(i) no nontrivial eigenvector

(ii) intersection: span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ ; Need $y$: $By = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$B \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = 0$

(iii) Null space outside intersection: $n = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ :

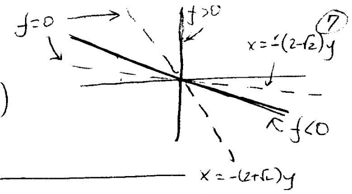
$$\text{i.e. } B \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{V} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{Jordan form}}$$

$$B = V \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^{-1}$$

6.1.1 $f = x^2 + 4xy + 2y^2 = (x+2y)^2 - 2y^2$

$= (x+(2+\sqrt{2})y)(x+(2-\sqrt{2})y)$

(2) Saddlepoint

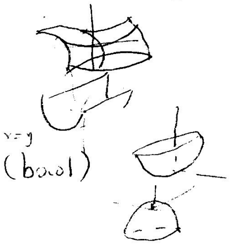



6.1.2 (a) $f = x^2 + 6xy + 5y^2 = (x+3y)^2 - 4y^2$ saddlepoint

$9-5>0$

(b) $f = x - 2xy + y^2 = (x-y)^2$ $(x=y)$ trough

(c) $f = 2x^2 + 6xy + 5y^2 = (\sqrt{2}x + \frac{3}{\sqrt{2}}y)^2 + \frac{1}{2}y^2$ definite (bowl)




(d) $f = -x^2 + 4xy - 8y^2 = -(x-2y)^2 - 4y^2$ neg. definite

negative diagonals, positive determinant.

6.1.3 let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $a>0$, $ac-b^2>0 \Rightarrow \underline{c>0}$

Then $\det(A-\lambda I) = (a-\lambda)(c-\lambda) - b^2 = \lambda^2 - (a+c)\lambda + ac - b^2 = 0$

$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} = \frac{(a+c) \pm \sqrt{(a-c)^2+4b^2}}{2}$

(i) Discriminant is positive $\Rightarrow 0 < (a+c)^2 - 4(ac-b^2) < (a+c)^2$

since $ac-b^2>0$. Thus $\lambda > 0$.

*6.1.4 (a) $F(0,0) = -1+4-3>0$

2

$DF|_{(0,0)} = (4(e^x-1) - 5\sin y,\ -5x\cos y + 12y)|_{(0,0)} = (0,0)$

$D^2F = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 4e^x & -5\cos y \\ -5\cos y & 12 \end{pmatrix}_{(0,0)} = \begin{pmatrix} 4 & -5 \\ -5 & 12 \end{pmatrix}$

The Hessian at (0,0) has positive diagonals and $\det = 48-25 = 23 > 0$

$\Rightarrow F$ has a minimum at (0,0).

(c) $F(0,0) = 0$ ; $DF|_{(0,0)} = ((2x-2)\cos y, -(x^2-2x)\sin y)|_{(1,\pi)} = (0,0)$

$D^2F = \begin{pmatrix} 2\cos y & -2(x-1)\sin y \\ -2(x-1)\sin y & -(x^2-2x)\cos y \end{pmatrix}_{(1,\pi)} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ negative definite: maximum at $(1,\pi)$ ($\det = 2 > 0$).

6.1.7 (a)

$$f_1 = x_1^2+x_2^2+x_3^2-2x_1x_2-2x_1x_3+2x_2x_3 = (x_1x_2x_3)\underbrace{\begin{pmatrix}1&-1&-1\\-1&1&1\\-1&1&1\end{pmatrix}}_{=A_1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$f_2 = x_1^2+2x_2^2+11x_3^2-2x_1x_2-2x_1x_3-4x_2x_3 = (x_1x_2x_3)\underbrace{\begin{pmatrix}1&-1&-1\\-1&2&-2\\-1&-2&11\end{pmatrix}}_{=A_2}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

(b) Show $f_1$ is a single perfect square and not pos. def. When is $f_1=0$?

We have $$L_1A_1 = \underbrace{\begin{pmatrix}1&&\\1&1&\\1&0&1\end{pmatrix}}_{L_1}\underbrace{\begin{pmatrix}1&-1&-1\\-1&1&1\\-1&1&1\end{pmatrix}}_{A_1} = \underbrace{\begin{pmatrix}1&-1&-1\\0&0&0\\0&0&0\end{pmatrix}}_{U_1} = \underbrace{\begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix}}_{D}\underbrace{\begin{pmatrix}1&-1&-1\\&1&0\\&&1\end{pmatrix}}_{L_1^T}$$

$$\Rightarrow A_1 = \begin{pmatrix}1&&\\-1&1&\\-1&0&1\end{pmatrix}\begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix}\begin{pmatrix}1&-1&-1\\&1&0\\&&1\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}(1\ -1\ -1)$$

i.e. $x^TA_1x = (x_1-x_2-x_3)^2 = 0$ if $x_1-x_2-x_3=0$

$\therefore$ There are two zero pivots $\Rightarrow$ not pos. def.

~~Alternatively, can do by diagonalizing~~

(c) Factor $A_2 = LL^T$; write $f_2$ as sum of 3 squares:

$$\begin{pmatrix}1&-1&-1\\-1&2&-2\\-1&-2&11\end{pmatrix}\longrightarrow\begin{pmatrix}1&-1&-1\\0&1&-3\\0&-3&10\end{pmatrix}\longrightarrow\begin{pmatrix}1&-1&-1\\0&1&-3\\0&0&1\end{pmatrix} = U = L^T$$

$$x^TA_2x = \left(x^T\begin{pmatrix}1&0&0\\-1&1&0\\-1&-3&1\end{pmatrix}\right)\left(\begin{pmatrix}1&-1&-1\\0&1&-3\\0&0&1\end{pmatrix}x\right)$$

$$= \cancel{x_1^2} \; (x_1-x_2-x_3)^2+(x_2-x_3)^2+x_3^2$$

**6.1.17** $A^{m\times n}$, rank $A=n \Rightarrow A^TA$ is $n\times n$, invertible. (4) (14)
Rewrite $x^TA^TAx$ to show why it is positive for $\vec{x}\neq 0$.
(This shows $A^TA$ is positive definite).

$x^TA^TAx = (Ax)^T(Ax) := y^Ty > 0$ if $y\neq 0$; since $A$ has ~~full~~ rank $n$, it has no null vectors $\Rightarrow Ax\neq 0$ if $x\neq 0$.
i.e. $x\neq 0 \Rightarrow Ax\neq 0 \Rightarrow y^Ty>0 \Leftrightarrow x^TA^TAx>0$ if $x\neq 0$.

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**6.2.7** If $A=Q\Lambda Q^T$ is symmetric positive definite, then $R=Q\sqrt{\Lambda}Q^T$ is its symmetric positive definite square root. Why does $R$ have positive eigenvalues? Compute $R$ & verify $R^2=A$ for $A_1=\begin{pmatrix}10&6\\6&10\end{pmatrix}$, $A_2=\begin{pmatrix}10&-6\\-6&10\end{pmatrix}$

Obviously, the numbers $\sqrt{\lambda_i}$, $i=1,\dots,n$ ($A$ is $n\times n$) are the eigenvalues and we chose them to be positive (in general, $A$ will have $2^n$ distinct square roots if we choose $\pm\sqrt{\lambda_i}$).

(4) $|A_i-\lambda I| = \begin{vmatrix}10-\lambda & \pm 6\\ \pm 6 & 10-\lambda\end{vmatrix} = (10-\lambda)^2-6^2 = (\lambda-16)(\lambda-4)$ ~~$\lambda^2-20\lambda$~~, $\lambda=4,16$
(same eigenvalues for both).

eigenvectors $\begin{pmatrix}10-\lambda & \pm 6\\ \pm 6 & 10-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\vec{0} \Rightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}\pm 6\\ \lambda-10\end{pmatrix}$ $\xrightarrow{A_1}$ $\lambda_1=4, q_1'=\begin{pmatrix}1\\-1\end{pmatrix}$, $\lambda_2=16, q_2'=\begin{pmatrix}1\\1\end{pmatrix}$

$A_1\begin{pmatrix}1/\sqrt2 & 1/\sqrt2\\ -1/\sqrt2 & 1/\sqrt2\end{pmatrix} = \begin{pmatrix}1/\sqrt2 & 1/\sqrt2\\ -1/\sqrt2 & 1/\sqrt2\end{pmatrix}\begin{pmatrix}4&0\\0&16\end{pmatrix}$, $\sqrt{\Lambda}=\begin{pmatrix}2&0\\0&4\end{pmatrix}$

$R_1=Q\sqrt{\Lambda}Q^T=\begin{pmatrix}1/\sqrt2 & 1/\sqrt2\\ -1/\sqrt2 & 1/\sqrt2\end{pmatrix}\begin{pmatrix}2&0\\0&4\end{pmatrix}\begin{pmatrix}1/\sqrt2 & -1/\sqrt2\\ 1/\sqrt2 & 1/\sqrt2\end{pmatrix}=\begin{pmatrix}3&1\\1&3\end{pmatrix}$; $\begin{pmatrix}3&1\\1&3\end{pmatrix}\begin{pmatrix}3&1\\1&3\end{pmatrix}=\begin{pmatrix}10&6\\6&10\end{pmatrix}$

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$A_2$: $q_1'=\begin{pmatrix}1\\1\end{pmatrix}$, $q_2'=\begin{pmatrix}1\\-1\end{pmatrix}$:

$R_2=\begin{pmatrix}1/\sqrt2 & 1/\sqrt2\\ 1/\sqrt2 & -1/\sqrt2\end{pmatrix}\begin{pmatrix}2&0\\0&4\end{pmatrix}\begin{pmatrix}1/\sqrt2 & 1/\sqrt2\\ 1/\sqrt2 & -1/\sqrt2\end{pmatrix}=\begin{pmatrix}3&-1\\-1&3\end{pmatrix}$; $\begin{pmatrix}3&-1\\-1&3\end{pmatrix}\begin{pmatrix}3&-1\\-1&3\end{pmatrix}=\begin{pmatrix}10&-6\\-6&10\end{pmatrix}$

**6.2.9** $A = R^T R$. Let $Rx = u$, $Ry = w$

By Cauchy-Schwartz: $|u^T w|^2 \leq (u^T u)(w^T w)$

$$\Rightarrow |x^T R^T R y|^2 \leq (x^T R^T R x)(y^T R^T R y)$$

$$\Rightarrow |x^T A y|^2 \leq (x^T A x)(y^T A y)$$

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**6.2.14** Determine definiteness of

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{pmatrix}$ : $d_1 = a_{11} = 1 > 0$; $\det A_2 = 5 - 4 = 1$ : $d_2 = 1 > 0$; $\det A_3 = -4 \Rightarrow d_3 = -4$ } A indefinite pivots

not ~~de~~semidefinite ; thus $x^T A x = -1$ has real solutions. let $y_3 = 1/2$

$x^T A x = d_1 y_1^2 + y_2^2 - 4y_3^2 = -1$ :

(can write: $x^T A x = d_1 y_1^2 + y_2^2 - 4y_3^2$ where $A = L D L^T$, $D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -4 \end{pmatrix}$ ↳ pivots, L found as usual

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$B = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$ $d_1 = b_{11} = 1$, $d_2 = \frac{\det B_2}{1} = 2/1 = 2$

$\det B_3 = 6$, $d_3 = 6/2 = 3$ ; $\det B = 10$, $d_4 = \frac{10}{6}$

all pivots positive $\Rightarrow$ B pos. def.

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$C = -B$ : now all pivots negative $\Rightarrow$ negative definite

$(-1, -2, -3, -5/3)$.

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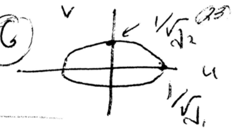
$D = A^{-1}$ : since A indefinite : 2 pivots $> 0$, 1 pivot $< 0$

$\Rightarrow$ eigenvalues likewise : $\lambda_1, \lambda_2 > 0$, $\lambda_3 < 0$

$\Rightarrow A^{-1}$ has evalues $1/\lambda_1, 1/\lambda_2 > 0$, $1/\lambda_3 < 0$.

i.e. indefinite

**6.2.10** $u^2+4v^2=1 \qquad \lambda_1=1,\ e_1=\begin{pmatrix}1\\0\end{pmatrix};\ \lambda_2=4,\ e_2=\begin{pmatrix}0\\1\end{pmatrix}$ (c)

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**2.11** $3u^2-2\sqrt{2}\,uv+2v^2=1 \Rightarrow (u\ v)\begin{pmatrix}3&-\sqrt2\\-\sqrt2&2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=1$

*

$$\begin{vmatrix}3-\lambda&-\sqrt2\\-\sqrt2&2-\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-5\lambda+4=0 \Rightarrow (\lambda-1)(\lambda-4)=0$$

$$\begin{pmatrix}3&-\sqrt2\\-\sqrt2&2\end{pmatrix}\begin{pmatrix}1/\sqrt3&\sqrt2/\sqrt3\\\sqrt2/\sqrt3&-1/\sqrt3\end{pmatrix}=\begin{pmatrix}1/\sqrt3&\sqrt2/\sqrt3\\\sqrt2/\sqrt3&-1/\sqrt3\end{pmatrix}\begin{pmatrix}1&0\\0&4\end{pmatrix}$$

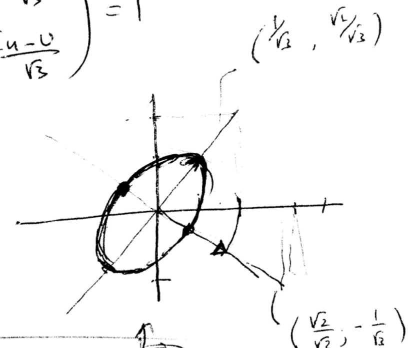
$$\begin{pmatrix}3&-\sqrt2\\-\sqrt2&2\end{pmatrix}=\frac13\begin{pmatrix}1&\sqrt2\\\sqrt2&-1\end{pmatrix}\begin{pmatrix}1&0\\0&4\end{pmatrix}\begin{pmatrix}1&\sqrt2\\\sqrt2&-1\end{pmatrix}$$

i.e. $(u,v)\begin{pmatrix}1/\sqrt3&\sqrt{2/3}\\\sqrt{2/3}&-1/\sqrt3\end{pmatrix}\begin{pmatrix}1&0\\0&4\end{pmatrix}\begin{pmatrix}1/\sqrt3&\sqrt{2/3}\\\sqrt{2/3}&-1/\sqrt3\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=1$

$$\Rightarrow \left(\begin{matrix}\frac{u+\sqrt2 v}{\sqrt3}\\\frac{\sqrt2 u-v}{\sqrt3}\end{matrix}\right.\quad\left(\frac{u+\sqrt2v}{\sqrt3}\quad\frac{\sqrt2u-v}{\sqrt3}\right)\begin{pmatrix}1&0\\0&4\end{pmatrix}\begin{pmatrix}\frac{u+\sqrt2v}{\sqrt3}\\\frac{\sqrt2u-v}{\sqrt3}\end{pmatrix}=1$$

$$\Rightarrow \left(\frac{u+\sqrt2v}{\sqrt3}\right)^2+4\left(\frac{\sqrt2u-v}{\sqrt3}\right)^2=1$$

orthogonal coords.

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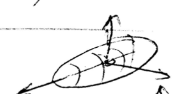
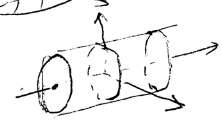
**6.2.12** 1) $\lambda_1,\lambda_2,\lambda_3>0$: triaxial ellipsoid



(2) $\lambda_1,\lambda_2>0,\ \lambda_3=0$: elliptic cylinder



(3) $\lambda_1>0,\ \lambda_2=\lambda_3=0$:

(4) $\lambda_1=\lambda_2=\lambda_3=0$ : all space.

$\boxed{6.2.13}$ $\det(-A) = -\det A$ $\quad \left( \det_{A\in\mathbb{R}^{n\times n}} \lambda A = \lambda^n \det A \right)$ ⑦

(I) $x^TAx \leqslant 0$, $=0$ iff $x=0$

(II) spectrum $A = \{\lambda_i\}_1^3$, $\lambda_i < 0$

(III) $a_{11} < 0$, $\det A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0$ $\quad \det A < 0$

(IV) $d_1 = a_{11} < 0$, $d_2 = \frac{\det A_2}{a_{11}} < 0$, $d_3 = \frac{\det A}{\det A_2} \cancel{>} 0$

(V) $\boxed{-A = R^TR}$

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$\boxed{6.2.14}$ ~~old~~ If a diagonal entry is zero $-A$ cannot be pos. def.

$\Rightarrow A$ cannot be neg. def.

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* $\boxed{6.2.15}$ New: 6.2.14 $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix}$

$d_1 = a_{11} > 0$; $\det A_2 = 1$, $d_2 = 1 > 0$; $\det A_3 = -4 \Rightarrow d_3 = -4$

not indefinite (thus, $x^TAx = -1$ has real solutions)

Since eigenvalues are also arranged similarly, $(2+, 1-)$,

then $A^{-1}$ is also non-definite.

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$B = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$ $\quad d_1 = b_{11} = 1$, $d_2 = \frac{\det B_2}{1} = \frac{2}{1} = 2$

~~$d_3$~~ $\det B_3 = 6$, $d_3 = \frac{\det B_3}{\det B_2} = \frac{6}{2} = 3$

$\det B_4 = 10$, $d_4 = \frac{10}{6} > 0$ pos. def

$-B$ negative definite.

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$\boxed{6.2.16}$ F : ($Q^TAQ$ not diagonal)

($A = A^T$, $Q^{-1} = Q^T$ def)

T : $Q^TAQ$ symm., p.def.

T : $Q^TAQ \sim A$ so same evalues

T : $e^{-A}$ is symm. p.def (evalues $e^{-\lambda_i} > 0$).

$\boxed{6.2.17}$ if $A$ pos. def., then $x^TAx > 0$

let $x^T = (0 \; x_2 \dots x_n) \Rightarrow \det A_{11} > 0$

↳ cofactor of $a_{11}$

but $\det A = a_{11}A_{11}$ + other terms. Since $a_{11} > 0$, $A_{11} > 0$

Then if $a_{11} \to a_{11} + \varepsilon$ then $\det A \to \det A' = \varepsilon A_{11} + \det A > \det A$.

If $A_{11} < 0$, then this fails, but then $A$ indefinite

(ex: $A = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$ ) $\det A = -7$ ; $A' = \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$, $\det A' = -9 < \det A$

$\boxed{6.2.18}$ (old) $A = R^TR \Rightarrow \det A = (\det R)^2 = \prod_{i=1}^n r_{ii}^2 = r_{11}^2 \cdots r_{nn}^2$

~~Now~~ $a_{ii}$ = using ~~a lower~~ on upper triangular $R$ (say from $LDL^T$ factorization), with $r_{ii}^2 = d_i$, ith pivot.

Now $a_{ii} = \sum_{k=1}^{i} r_{ki}^2 \geq r_{ii}^2$

Thus $\det A = \prod_{i=1}^n d_i = \prod_{i=1}^n r_{ii}^2 \leq \prod_{i=1}^n \left( \sum_{k=1}^{i} r_{ki}^2 \right) = \prod_{i=1}^n a_i$

* $\boxed{6.2.19}$ New 6.2.18 $AM + M^HA = -I$, $A$ pos.def. ; let $Mx = \lambda x$ ; then

$x^HAMx + x^HM^HAx = -\|x\|^2 \Rightarrow \lambda(x^HAx) + \bar{\lambda}(x^HAx) = -\|x\|^2$

$\Rightarrow 2 \,\mathrm{Re}\,\lambda (x^HAx) = -\|x\|^2 < 0$ ; but $x^HAx > 0$

$\Rightarrow \boxed{\mathrm{Re}\,\lambda < 0}$

* 6.2.27 With positive pivots in D, the factorization ~~LD~~ 9

$A = LDL^T$ becomes $L\sqrt{D}\sqrt{D}L^T$. Then $C = L\sqrt{D}$ yields the Cholesky factorization $A = CC^T$ (symmetrized LU)

(a) From $C = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ find $A = CC^T = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 3 & 5 \end{pmatrix}$

(b) From $A = {}_{-2}\begin{pmatrix} 4 & 8 \\ 8 & 25 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 8 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \underbrace{\left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\right]}_{= C}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$$

* 6.2.28 Find C so $A = CC^T$ for: (C lower triangular)

~~Note 6.2.27~~

$$A = {}_{-2}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 9 & & \\ & 1 & \\ & & 4 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

$$C = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 3 & & \\ & 1 & \\ & & 2 \end{pmatrix}}_{\sqrt{D}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

$$A = \begin{matrix} \\ -1 \\ -1 \end{matrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{pmatrix} \rightarrow \begin{matrix} \\ \\ -1 \end{matrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & \sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \sqrt{5} \end{pmatrix}$$