p. 262, 5.3.4 $\qquad G_{k+2} = \frac{1}{2}\left(G_{k+1} + G_k\right)$

$$\begin{pmatrix} G_{k+2} \\ G_{k+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix} \qquad G_0 = 0, \; G_1 = \frac{1}{2}$$

$$\begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \frac{1}{2}\lambda - 1/2 = 0 \Rightarrow \lambda = \frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} + 2}$$

$$\lambda_\pm = \frac{1}{4} \pm \frac{3}{4} \begin{cases} \to 1 \\ \to -1/2 \end{cases}$$

$$\lambda = 1; \quad \underline{e}_1 = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad \lambda = -\frac{1}{2}, \quad \underline{e}_{-1/2} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

So:
$$\begin{pmatrix} 1/2 & 1/2 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{1 + 1/2} \begin{pmatrix} 1 & 1/2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{pmatrix} = X^{-1}$$

$$A = X \Lambda X^{-1} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{pmatrix}$$

$$A^k = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1/2)^k \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{pmatrix}$$

So
$$\begin{pmatrix} G_k \\ G_{k-1} \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1/2)^{k-1} \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1/2)^{k-1} \end{pmatrix} \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -\left(-\frac{1}{2}\right)^{k-1} \end{pmatrix}$$

$$\Rightarrow \quad G_k = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{k}\right) \xrightarrow{k \to \infty} \frac{1}{3}$$

p. 2~~62~~, 5.3.B

$$Z_{k+1} = A z_k \; ; \quad A = \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 1, \; \lambda_2 = 3/4, \; \lambda_3 = 1/2$

Eigenvectors:

$\lambda_1 = 1$: $\begin{pmatrix} 0 & 1/4 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \underline{0} \Rightarrow \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 3/4$: $\begin{pmatrix} 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & -1/4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = 0 \Rightarrow \underline{e}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda_3 = 1/2$: $\begin{pmatrix} 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = 0 \Rightarrow \underline{e}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

So $A = X \begin{pmatrix} 1 & & \\ & 3/4 & \\ & & 1/2 \end{pmatrix} X^{-1}$

$\Rightarrow A^n = X \begin{pmatrix} 1 & & \\ & (3/4)^n & \\ & & (1/2)^n \end{pmatrix} X^{-1}$

As $n \to \infty$, $\Lambda^n \to \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$

and $A^n \to X \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} X^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} X^{-1}$

Then, if initially $z = c_1 \underline{e}_1 + c_2 \underline{e}_2 + c_3 \underline{e}_3$

we have $A^n z = c_1 \lambda_1^n \underline{e}_1 + c_2 \lambda_2^n \underline{e}_2 + c_3 \lambda_3^n \underline{e}_3$

Since $\lambda_2^n ; \lambda_3^n \to 0$, $\lambda_1^n = 1$, $A^n z \to c_1 \underline{e}_1$

(i.e. eventually they all become dead!)

$\dfrac{5.3.16}{p.264}$ $\quad \dfrac{du}{dt} = Au = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u \Rightarrow u = e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} u_0$

$$\boxed{\underline{u}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \underline{u}_0}$$

(i) $u_{n+1} - u_n = A u_n \Rightarrow u_{n+1} = (I+A) u_n$ ~~backw~~ forward differencing

(ii) $u_{n+1} - u_n = A u_{n+1}$: $u_{n+1} = (I-A)^{-1} u_n$ backward

(iii) $u_{n+1} - u_n = \frac{1}{2} A(u_{n+1} + u_n) \Rightarrow u_{n+1} = (I - \frac{1}{2}A)^{-1}(I + \frac{1}{2}A) u_n$ (centered)

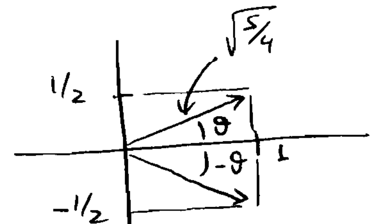
A : eigenvalues $(\pm i)$

$I + A$: eigenvalues $(1 \pm i)$

$I - A$: " $(1 \mp i) \Rightarrow (I-A)^{-1}$ evalues $\dfrac{1}{1 \mp i} = \dfrac{1}{\sqrt{2}} e^{\pm i\pi/4}$

$\left(\dfrac{1 \pm i}{\sqrt{2}} = e^{\pm i\pi/4} \right)$ (recal : $f(A)$ has evals. $f(\lambda_i)$)

$(I - \frac{1}{2}A)^{-1}(I + \frac{1}{2}A)$: evals: $\dfrac{1 \pm \frac{1}{2}i}{1 \mp \frac{1}{2}i}$



Now $\left(1 \pm \frac{1}{2}i\right) = \sqrt{\frac{5}{4}} e^{\pm i\vartheta} = \frac{1}{2}\sqrt{5}\, e^{\pm i\vartheta}$, $\vartheta = \tan^{-1}(\frac{1}{2})$

$\vartheta = \tan^{-1}(1/2)$

So $\dfrac{1 \pm \frac{1}{2}i}{1 \mp \frac{1}{2}i} = e^{(\pm i\vartheta) - (\mp i\vartheta)} = e^{\pm 2i\vartheta}$

Thus, the matrix $(I - \frac{1}{2}A)^{-1}(I + \frac{1}{2}A)$ is orthogonal and does not alter the norm of the iterate $\underline{u}_n$: the solutions for the centered difference scheme "stay on the circle".

5.3.10
p. 265

$(I-A)(I+A+A^2+\cdots) =$

$= (I+A+A^2+\cdots) - (A+A^2+A^3+\cdots) = \lim_{n\to\infty} I - A^n = I$

provided $A^n \xrightarrow{n\to\infty} 0$, i.e. provided $|\lambda_i| < 1$, $\lambda_i \in sp(A)$.

Ex: $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(A is "nil-potent")

i.e. $I + A + \cdots = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

while $I - A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

so $\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ & 1 & -1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; so, indeed,

$I + A + A^2 + \cdots = I + A + A^2 = (I-A)^{-1}$

---

i.e. $e^{A+B} = \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix} \neq e^A e^B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$= e^B e^A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

$\boxed{5.4.14}$ $\dfrac{d^2 u}{dt^2} = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix} u$

let $\underline{u} = \begin{pmatrix} a \\ b \end{pmatrix} e^{i\omega t}$; $-\omega^2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -5+\omega^2 & 4 \\ 4 & -5+\omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $(\omega^2-5)^2 - 16 = 0$

$\omega^2 = 5 \pm 4 = \begin{cases} 9 \\ 1 \end{cases}$

$\omega_1 = \pm 3$; $\omega_2 = \pm 1$

w8/1/7 Then: $\omega_1 = \pm 3$; $\begin{pmatrix} -5+9 & 4 \\ 4 & -5+9 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{e}_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\omega_1 = \pm 1$ $\begin{pmatrix} -5+1 & 4 \\ 4 & -5+1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So: $\boxed{\underline{u}(t) = (a_1 \cos 3t + b_1 \sin 3t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (a_2 \cos t + b_2 \sin t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\boxed{p.301, 5.5.7}$ $A = \begin{pmatrix} 1 & i & 0 \\ i & 0 & 1 \end{pmatrix}$; $A^H = \begin{pmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{pmatrix}$

$C = A^H A = \begin{pmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \\ i & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{pmatrix} = C^H$

$(C^H = (A^H A)^H = A^H (A^H)^H = A^H A = C)$.

$\boxed{5.5.8}$ $Ax = 0$:

$$-i\begin{pmatrix} 1 & i & 0 \\ i & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}:$$

$$\hookrightarrow \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-i} \longrightarrow \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 1 \end{pmatrix}$$

Null vector: $\underline{u} = \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$; $N(A) = \text{span}\left\{\begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}\right\}$

$$\mathcal{R}(A^H) = \text{span}\left\{ \underset{\underline{a}_1}{\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}}, \underset{\underline{a}_2}{\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}} \right\}$$

$$\left.\begin{aligned} \langle \underline{a}_1, \underline{u} \rangle &= a_1^H \underline{u} = (1 \;\; i \;\; 0)\begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix} = i - i = 0 \\ \langle \underline{a}_2, \underline{u} \rangle &= a_2^H \underline{u} = (i \;\; 0 \;\; 1)\begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix} = -1 + 1 = 0 \end{aligned}\right\}$$

$\boxed{5.5.11}$ $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$; $P - \lambda I$: $\begin{vmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4}$

$$\Rightarrow \left(\lambda - \frac{1}{2}\right) \pm \frac{1}{2} = 0 \Rightarrow \lambda = \begin{cases} 0 \\ +1 \end{cases}$$

$$\begin{pmatrix} 1/2-\lambda & 1/2 \\ & \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 : \quad \underline{e}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \underline{e}_{+1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = +1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \left(1/\sqrt{2} \;\; 1/\sqrt{2}\right)$$ (recall: need orthonormal vectors for spectral decomposition)

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 : \quad \lambda = \pm 1$$

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \underline{e}_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} :$$

$$\boxed{Q = \left(1/\sqrt{2} \;\; 1/\sqrt{2}\right)^T \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}^T - \left(1/\sqrt{2} \;\; -1/\sqrt{2}\right)^T \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}^T}$$

(error in order of these!)

$$R = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \; ; \; R - \lambda I = \begin{pmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix}$$

$$(\lambda+3)(\lambda-3) - 16 = 0 \Rightarrow \lambda^2 - 25 = 0 \Rightarrow \boxed{\lambda = \pm 5}$$

$$\underline{e}_5 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \; ; \; \underline{e}_{-5} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$R = 5 \begin{pmatrix} 2/\sqrt5 \\ 1/\sqrt5 \end{pmatrix} \begin{pmatrix} 2/\sqrt5 & 1/\sqrt5 \end{pmatrix} - 5 \begin{pmatrix} 1/\sqrt5 \\ -2/\sqrt5 \end{pmatrix} \begin{pmatrix} 1/\sqrt5 & -2/\sqrt5 \end{pmatrix}$$

5.5.19

$$\underline{u}_1 \times \underline{u}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1/\sqrt3 & 1/\sqrt3 & i/\sqrt3 \\ i/\sqrt2 & 0 & 1/\sqrt2 \end{vmatrix} = u_3^H$$

$$= \hat{i}\left(1/\sqrt6\right) - \hat{j}\left(1/\sqrt6 + 1/\sqrt6\right) + \hat{k}\left(-i/\sqrt6\right)$$

$$\underline{u}_3 = \left(1/\sqrt6 \, , \frac{-2}{\sqrt6} \, , \frac{-i}{\sqrt6}\right)^H = \begin{pmatrix} 1/\sqrt6 \\ \frac{-2}{\sqrt6} \\ i/\sqrt6 \end{pmatrix}$$

Then $u_3^H u_3 = \left(\frac{1}{6} + \frac{4}{6} + \frac{1}{6}\right) = 1$

$$u_1^H u_2 = \begin{pmatrix} 1/\sqrt3 & 1/\sqrt3 & -i/\sqrt3 \end{pmatrix} \begin{pmatrix} i/\sqrt2 \\ 0 \\ 1/\sqrt2 \end{pmatrix} = i/\sqrt6 - i/\sqrt6 = 0$$

$$u_1^H u_3 = \begin{pmatrix} 1/\sqrt3 & 1/\sqrt3 & -i/\sqrt3 \end{pmatrix} \begin{pmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ i/\sqrt6 \end{pmatrix} = \frac{1}{3\sqrt2} - \frac{2}{3\sqrt2} + \frac{1}{3\sqrt2} = 0$$

$$u_2^H u_3 = \begin{pmatrix} -i/\sqrt2 & 0 & 1/\sqrt2 \end{pmatrix} \begin{pmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ i/\sqrt6 \end{pmatrix} = \frac{-i}{2\sqrt3} + \frac{i}{2\sqrt3} = 0$$

* $\pm u_3$ will work equally well. (or, in general, $e^{i\vartheta}\vec{u}_3$, ϑ arbitrary)