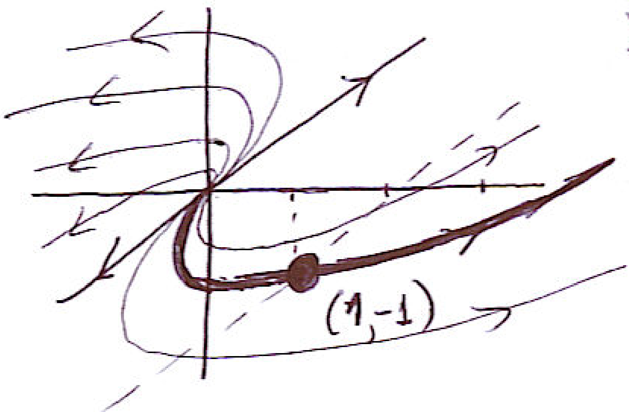
Prep. Exam 1 Solutions

(1.) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$: $\left.\begin{matrix} \mathrm{Tr} A = 2 \\ \det A = 1 \end{matrix}\right\}$ $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0$,

$\lambda = 1$ double

$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A - \lambda I)\underline{x} = 0 \Rightarrow \begin{pmatrix} 3-1 & -2 \\ 2 & -1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{0} \Rightarrow \begin{matrix} 2x_1 - 2x_2 = 0 \\ x_1 = 1,\ x_2 = 1 \end{matrix}$



$\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $(A - \lambda I)\,\underline{v} = \underline{x}_1 \Rightarrow 2v_1 - 2v_2 = 1 \Rightarrow v_1 = 1/2,\ v_2 = 0$

$$\underline{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right]e^t$$

$$\underline{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c_1 + \frac{1}{2}c_2 = 1 \\ c_1 = -1 \Rightarrow c_2 = 4 \end{matrix}$$

$$\underline{x}(t) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + 4\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}t + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right]e^t = \begin{pmatrix} 4t+1 \\ 4t-1 \end{pmatrix}e^t$$

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(2) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 6 & 3 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\begin{matrix} \mathrm{Tr} A = 8 \\ \det A = -9 \end{matrix}$ $\lambda^2 - 8\lambda - 9 = 0 \Rightarrow \lambda = 4 \pm \sqrt{16+9} = 4 \pm 5 = \begin{cases} 9 \\ -1 \end{cases}$

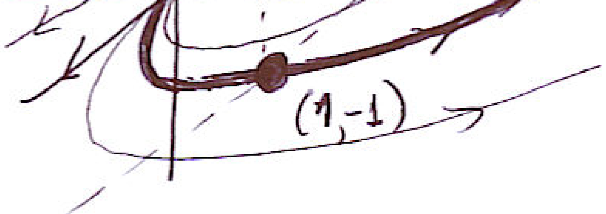
$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$(6-\lambda)x_1 + 3x_2 = 0 \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = \frac{\lambda - 6}{3} \end{cases}$ $\lambda_1 = -1,\ \underline{x}_1 = \begin{pmatrix} 1 \\ -\frac{7}{3} \end{pmatrix} \sim \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

$\underline{x}$.

$\lambda_2 = 9,\ \underline{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\underline{x}(t) = c_1 e^{-t}\begin{pmatrix} 3 \\ -7 \end{pmatrix} + c_2 e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$$\underline{X}(0) = \begin{pmatrix}1\\-1\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1/2\\0\end{pmatrix} \Rightarrow \begin{matrix} c_1 + \frac{1}{2}c_2 = 1 \\ c_1 = -1 \Rightarrow c_2 = 4\end{matrix}$$

$$\underline{X}(t) = -\begin{pmatrix}1\\1\end{pmatrix}e^t + 4\left[\begin{pmatrix}1\\1\end{pmatrix}t + \begin{pmatrix}1/2\\0\end{pmatrix}\right]e^t = \begin{pmatrix}4t+1\\4t-1\end{pmatrix}e^t$$

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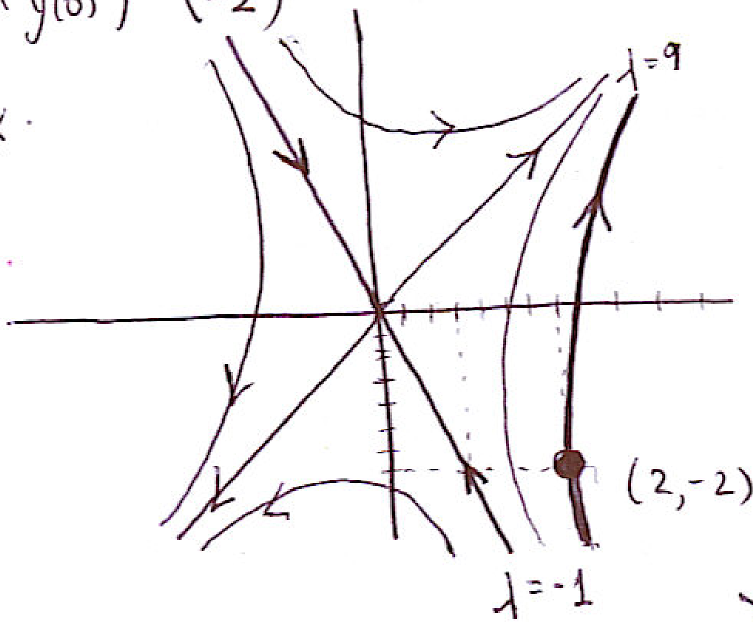
(2) $\begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}6 & 3\\7 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ $\quad$ $\mathrm{Tr}A = 8$, $\det A = -9$ $\quad$ $\lambda^2 - 8\lambda - 9 = 0 \Rightarrow \lambda = 4 \pm \sqrt{16+9} = 4 \pm 5 = \begin{cases}9\\-1\end{cases}$

$\begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$ $\quad$ $(6-\lambda)x_1 + 3x_2 = 0 \Rightarrow \begin{cases}x_1 = 1\\ x_2 = \frac{\lambda-6}{3}\end{cases}$ $\quad$ $\lambda_1 = -1,\ \underline{X}_1 = \begin{pmatrix}1\\-7/3\end{pmatrix} \sim \begin{pmatrix}3\\-7\end{pmatrix}$

$\underline{X}$.



$\lambda_2 = 9,\ \underline{X}_2 = \begin{pmatrix}1\\1\end{pmatrix}$

$$\underline{X}(t) = c_1 e^{-t}\begin{pmatrix}3\\-7\end{pmatrix} + c_2 e^{9t}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\underline{X}(0) = \begin{pmatrix}2\\-2\end{pmatrix} = c_1\begin{pmatrix}3\\-7\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}$$

$$\left.\begin{matrix}3c_1 + c_2 = 2\\ -7c_1 + c_2 = -2\end{matrix}\right. \Rightarrow \left.\begin{matrix}c_1 = 2/5\\ c_2 = 4/5\end{matrix}\right\}$$

$$\underline{X}(t) = \begin{pmatrix}\frac{6}{5}e^{-t} + \frac{4}{5}e^{9t}\\ -\frac{14}{5}e^{-t} + \frac{4}{5}e^{9t}\end{pmatrix}$$

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(3)(a) $\begin{pmatrix}x\\y\end{pmatrix}' = A\begin{pmatrix}x\\y\end{pmatrix}$; find type and stability of point at the origin:

(a) $A = \begin{pmatrix}5 & -1\\ \dots\end{pmatrix}$ | (b) $A = \begin{pmatrix}1 & -4\\ \dots\end{pmatrix}$ | (c) $A = \begin{pmatrix}3 & -2\\4 & -1\end{pmatrix}$ | (d) $A = \begin{pmatrix}2 & -5\\1 & -2\end{pmatrix}$

$$\underline{X}(0) = \begin{pmatrix}2\\-2\end{pmatrix} = c_1\begin{pmatrix}3\\-7\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}$$

$$\left.\begin{aligned} 3c_1 + c_2 &= 2 \\ -7c_1 + c_2 &= -2 \end{aligned}\right. \Rightarrow \left.\begin{aligned} c_1 &= 2/5 \\ c_2 &= 4/5 \end{aligned}\right\}$$

$$\underline{X}(t) = \begin{pmatrix} 6/5\, e^{-t} + 4/5\, e^{9t} \\ -14/5\, e^{-t} + 4/5\, e^{9t} \end{pmatrix}$$

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(3) (a) $\begin{pmatrix}x\\y\end{pmatrix}' = A\begin{pmatrix}x\\y\end{pmatrix}$; find type and stability of point at the origin:

(a) $A = \begin{pmatrix}5 & -1\\3 & 1\end{pmatrix}$

$\mathrm{Tr}A = 6$, $\det A = 8$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda_1 = -2$, $\lambda_2 = -4$

Stable node

(b) $A = \begin{pmatrix}1 & -4\\4 & -7\end{pmatrix}$

$\mathrm{Tr}A = -6$, $\det A = 9$

$\lambda^2 - 6\lambda + 9 = 0$

$\lambda = -3$, double

stable improper node

(c) $A = \begin{pmatrix}3 & -2\\4 & -1\end{pmatrix}$

$\mathrm{Tr}A = 2$, $\det A = 5$

$\lambda^2 - 2\lambda + 5 = 0$

$(\lambda-1)^2 + 4 = 0$

$\lambda = 1 \pm 2i$

unstable spiral

(d) $A = \begin{pmatrix}2 & -5\\1 & -2\end{pmatrix}$

$\mathrm{Tr}A = 0$, $\det A = 1$

$\lambda = \pm i$

centre, neutral (bull's eye)

(4) $y' - y = 1 + 3\cos t \Rightarrow (\phi y)' = (1+3\cos t)\phi$

where $\phi = e^{-\int dt} = e^{-t} \Rightarrow e^{-t} y = \int (1+3\cos t)e^{-t}\,dt + C$

$\Rightarrow e^{-t} y = 1 \cdot -e^{-t} + (A\cos t + B\sin t)e^{-t} + C$

where $\left((A\cos t + B\sin t)e^{-t}\right)' = 3\cos t\, e^{-t}$

$\Downarrow$

$-A\sin t\, e^{-t} - A\cos t\, e^{-t} + B\cos t\, e^{-t} - B\sin t\, e^{-t} = 3\cos t\, e^{-t}$

$\Rightarrow \begin{array}{ll} -A + B = 3 & (\cos t) \\ -A - B = 0 & (\sin t) \end{array} \Rightarrow \begin{array}{l} A = -3/2 \\ B = 3/2 \end{array}$

$y(t) = -1 - 3/2 \cos t + 3/2 \sin t + Ce^{t} \Rightarrow y(0) = y_0 = -1 - 3/2 + C$

$\Rightarrow C = y_0 + 5/2$ i.e. $y(t) = -1 - 3/2(\cos t - \sin t) + (y_0 + 5/2)e^{t}$

To stay bounded at infinity ($t \to +\infty$) need $y_0 = -5/2$ so exp. growing part vanishes.

(5.) $y' = xy^2(1+x^2)^{-1/2} \Rightarrow \frac{1}{y^2}y' = \frac{x}{\sqrt{1+x^2}} \Rightarrow$

$\frac{d}{dx}\left(-\frac{1}{y}\right) = \frac{d}{dx}\left(\sqrt{1+x^2}\right) \Rightarrow -\frac{1}{y} = \sqrt{1+x^2} + C$

$\Rightarrow y(x) = \frac{1}{C+\sqrt{1+x^2}}$; $y(0) = 1 \Rightarrow C = 0$

$\Rightarrow y(x) = (1+x^2)^{-1/2}$; exists for $-\infty < x < \infty$

(replaces soln. to problem (5) posted originally it was wrong.

(3.)(a) $\underline{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \underline{x} \Rightarrow \lambda^2 - 6\lambda + 8 = 0 \Rightarrow (\lambda-2)(\lambda-4) = 0$

Tr = 6, det = 8

$\Rightarrow \lambda = 2, 4$: $\left\{ \begin{array}{l} \text{unstable} \\ \text{node} \end{array} \right.$

(Note that signs are different than shown in prob. (3a)

The ones here are correct!