Solutions, 316-VIII

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1 Problem 4.3.12

Verify independence, give general solution and solve IVP for:

$$y'' - y = 0 \ ; \ y_1(x) = \cosh x \ , \ y_2(x) = \sinh x$$

$$y(0) = 1 \ , \ y'(0) = -1 \ .$$

Solution:
We compute the Wronskian:

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x = 1$$

Also, both functions satisfy the ODE since $(\cosh x)'' = \cosh x$, $(\sinh x)'' = \sinh x$ so that $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions. The general solution is therefore

$$y_g(x) = A\cosh x + B\sinh x$$

and

$$y_g'(x) = A\sinh x + B\cosh x$$

so

$$\begin{aligned} y_g(0) &= A\cosh 0 + B\sinh 0 \quad \Rightarrow \quad A = 1 \\ y_g'(0) &= A\sinh 0 + B\cosh 0 \quad \Rightarrow \quad B = -1 \end{aligned}$$

The solution to the IVP is

$$y(t) = \cosh x - \sinh x \ .$$

2 Problem 4.3.19

Use Abel's identity to determine the Wronskian of two solutions of

$$xy'' + (x-1)y' + 3y = 0 \ .$$

Solution:
First write equation into normal form $y'' + p(x)y' + q(x)y = 0$:

$$y'' + \frac{x-1}{x}y' + \frac{3}{x}y = 0$$

so that

$$p(x) = \frac{x-1}{x}$$

and

$$\int p(x)dx = \int \frac{x-1}{x}dx = x - \log|x| \ .$$

Then

$$W\left[y_1, y_2\right](x) = Ce^{-\int p(x)dx} = Cxe^{-x} \ .$$

3 Problem 4.3.23

Show that the substitution $y(x) = v(x)u(x)$ where

$$u(x) = e^{-\frac{1}{2}\int p(x)dx} ,$$

transforms the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

into an equation of the form:

$$u'' + f(x)u = 0 .$$

Solution:

$$\begin{aligned} y &= v(x)u(x) \\ y' &= v(x)u'(x) + v'(x)u(x) \\ y'' &= v(x)u''(x) + 2v'(x)u'(x) + v''(x)u(x) \end{aligned}$$

and substituting for $u(x)$

$$\begin{aligned} q(x)y &= q(x)v(x)e^{-\frac{1}{2}\int p(x)dx} \\ p(x)y' &= p(x)\left(-\frac{1}{2}v(x)p(x) + v'(x)\right)e^{-\frac{1}{2}\int p(x)dx} \\ y'' &= \left(-\frac{1}{2}v'(x)p(x) - \frac{1}{2}v(x)p'(x) + v''(x) + \frac{1}{4}v(x)p^2(x) - \frac{1}{2}v'(x)p(x)\right)e^{-\frac{1}{2}\int p(x)dx} \end{aligned}$$

and combining we see that the term involving $v'(x)$ cancels and we get

$$\left(v''(x) + \left[q(x) - \frac{1}{4}p^2(x) - \frac{1}{2}p'(x)\right]v(x)\right)e^{-\frac{1}{2}\int p(x)dx} = 0 .$$

This is of the desired form with

$$f(x) = q(x) - \frac{1}{4}p^2(x) - \frac{1}{2}p'(x) .$$

4 Problem 4.4.5

Find a second solution for

$$tx'' - (t+1)x' + x = 0 \ , \ t > 0$$

given a solution $x_1(1) = exp(t)$.
Solution:
The equation in canonical form is

$$x'' - \frac{t+1}{t}x' + \frac{1}{t}x = 0 \ , \ t > 0$$

so that

$$p(t) = -\frac{t+1}{t}$$

and

$$\int p(x)dx = -t - \log|t| \Rightarrow e^{-\int p(t)dt} = te^t$$

A second solution can be found in the form

$$x_2(t) = e^t \int \frac{e^{-\int p(t)dt}}{e^{2t}} dt = e^t \int \frac{te^t}{e^{2t}} dt$$

$$x_2(t) = e^t \int te^{-t} dt = -e^t(t+1)e^{-t} = -(t+1) \ .$$

5 Problem 4.4.13b

Consider Hermite's eq.

$$y'' - 2xy' + \lambda y = 0 \ ,$$

with λ a parameter. Give a second solution if we have $y_1(x) = f(x) = 3x - 2x^3$ for $\lambda = 6$.

Solution:

We have that

$$p(x) = -2x \Rightarrow e^{-\int p(x)dx} = e^{x^2}$$

and

$$y_2(t) = (3x - 2x^3) \int \frac{e^{-\int p(t)dt}}{(3x - 2x^3)^2} dx = (3x - 2x^3) \int \frac{e^{x^2}}{(3x - 2x^3)^2} dx \ .$$

6 Problem 4.4.14b

Find a second solution of Legendre's equation

$$(1 - x^2)y'' - 2xy' + \lambda(\lambda + 1)y = 0 \ , \ -1 < x < 1 \ ,$$

given

$$y_1(x) = f(x) = 3x^2 - 1 \ , \ \lambda = 2 \ .$$

Solution:

The equation in canonical form is

$$y'' - \frac{2x}{(1 - x^2)} y' + \frac{\lambda(\lambda + 1)}{(1 - x^2)} y = 0$$

so that

$$p(x) = -\frac{2x}{(1 - x^2)} \Rightarrow e^{-\int p(x)dx} = (1 - x^2)$$

and

$$y_2(t) = (3x - 2x^2) \int \frac{e^{-\int p(t)dt}}{(3x - 2x^2)^2} dx = (3x^2 - 1) \int \frac{1 - x^2}{(3x^2 - 1)^2} dx \ .$$