Solutions, 316-V

February 5, 2003

1 Problem 4.1.1

Solve the ODE by assuming a solution in the form $A\cos\omega t + B\sin\omega t$:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 4y = 3\sin 5t\ .$$

Solution:
We substitute (with $\omega = 5$):

$$\begin{aligned} y(t) &= A\cos 5t + B\sin 5t \\ y'(t) &= -5A\sin 5t + 5B\cos 5t \\ y''(t) &= -25A\cos 5t - 25B\sin 5t \end{aligned}$$

into the ODE:

$$\begin{array}{rcl} 4y & = & 4A\cos 5t + 4B\sin 5t \\ + & & + \\ 2\dfrac{dy}{dt} & = & -10A\sin 5t + 10B\cos 5t \\ + & & + \\ \dfrac{d^2y}{dt^2} & = & -25A\cos 5t - 25B\sin 5t \\ = & & = \\ 3\sin 5t & = & (-21A + 10B)\cos 5t + (-10A - 21B)\sin 5t \end{array}$$

which can be true if, and only if,

$$\begin{array}{rrcl} -21A & +10B & = & 0 \\ -10A & -21B & = & 3 \end{array}$$

Solving this system, find $A = -30/541$, $B = -63/541$, i.e.

$$y(t) = -30/541 \cos 5t - 63/541 \sin 5t \ .$$

2 Problem 4.1.4

Discussion of Resonance: consider the forced/damped system:

$$my'' + by' + ky = \cos \Omega t \ .$$

1. Find the synchronous solution $y(t) = A \cos \Omega t + B \sin \Omega t$
2. Sketch graphs of the coefficients A, B as functions of Ω for $m = 1$, $b = 0.1$, $k = 25$.

Solution:

1. We substitute:

$$\begin{aligned} y(t) &= A \cos \Omega t + B \sin \Omega t \\ y'(t) &= -\Omega A \sin \Omega t + \Omega B \cos \Omega t \\ y''(t) &= -\Omega^2 A \cos \Omega t - \Omega^2 B \sin \Omega t \end{aligned}$$

into the ODE:

$$\begin{array}{ccc} ky & = & kA \cos \Omega t + kB \sin \Omega t \\ + & & + \\ b\dfrac{dy}{dt} & = & -b\Omega A \sin \Omega t + b\Omega B \cos \Omega t \\ + & & + \\ m\dfrac{d^2y}{dt^2} & = & -m\Omega^2 A \cos \Omega t - m\Omega^2 B \sin \Omega t \\ = & & = \\ \cos \Omega t & = & ((k - m\Omega^2)A + b\Omega B) \cos \Omega t + (-b\Omega A + (k - m\Omega^2)B) \sin \Omega t \end{array}$$

We now set the coefficients of $\sin \Omega t$ and $\cos \Omega t$ on both sides of the equation equal to each other, to get the system:

$$\begin{array}{rrcl} (k - m\Omega^2)A & +b\Omega B & = & 1 \\ -b\Omega A & +(k - m\Omega^2)B & = & 0 \end{array}$$

Solving this system we find:

$$A = \frac{k - m\Omega^2}{(k - m\Omega^2)^2 + (b\Omega)^2}$$

$$B = \frac{b\Omega}{(k - m\Omega^2)^2 + (b\Omega)^2}$$

Then, the solution is:

$$y(t) = \frac{(k - m\Omega^2)\sin\Omega t + b\Omega\cos\Omega t}{(k - m\Omega^2)^2 + (b\Omega)^2}$$

2. We now plot A, B as functions of Ω; we give the plot using both Maple and Matlab.

   (a) **MATLAB**:

```
function rescoeff(m,b,k)
% plots the forced vibration coefficients A,B as functions
% of the forcing frequency Omega
Om(1:202) = 0; dOm = 2*sqrt(k/m)/200;
for i = 1:201
a1 = Om(i)*b;
Om2 = Om(i)*Om(i);
a2 = k - m*Om2;
denom = a1*a1 + a2*a2;
A(i) = a2 / denom;
B(i) = a1 / denom;
Om(i+1) = Om(i) + dOm;
end
subplot(2,1,1)
plot(Om(1:201),A,'r-')
title(sprintf('graph of A vs. Omega, with Omega0= %6.2f', sqrt(k/m)))
subplot(2,1,2)
plot(Om(1:201),B,'b-')
title(sprintf('graph of B vs. Omega, with Omega0= %6.2f', sqrt(k/m)))
```

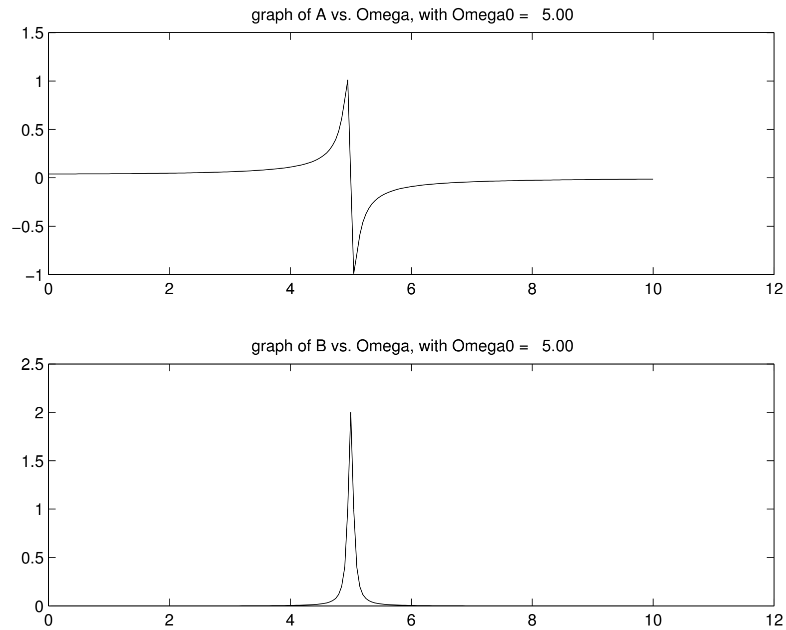


Figure 1. Plot of coefficients A, B of

$$y(t) = A\cos(\Omega * t) + B\sin(\Omega * t)$$

as functions of the forcing frequency, Ω.
Here y satisfies the forced-damped oscillator equation

$$my'' + by' + ky = \cos(\Omega * t)$$

for $m = 1, b = .1, k = 25$.
Note the sudden jumps in values as the natural frequency, $\Omega_0 = \sqrt{\frac{k}{m}} = 5$, is crossed (By Matlab)

(b) **MAPLE**

```
> restart:
> k:=25;
```

$$k := 25$$

```
> b:=.1;
```

$$b := .1$$

```
> m:=1;
```

$$m := 1$$

```
> Om0 := evalf(sqrt(k/m));
```

$$Om0 := 5.$$

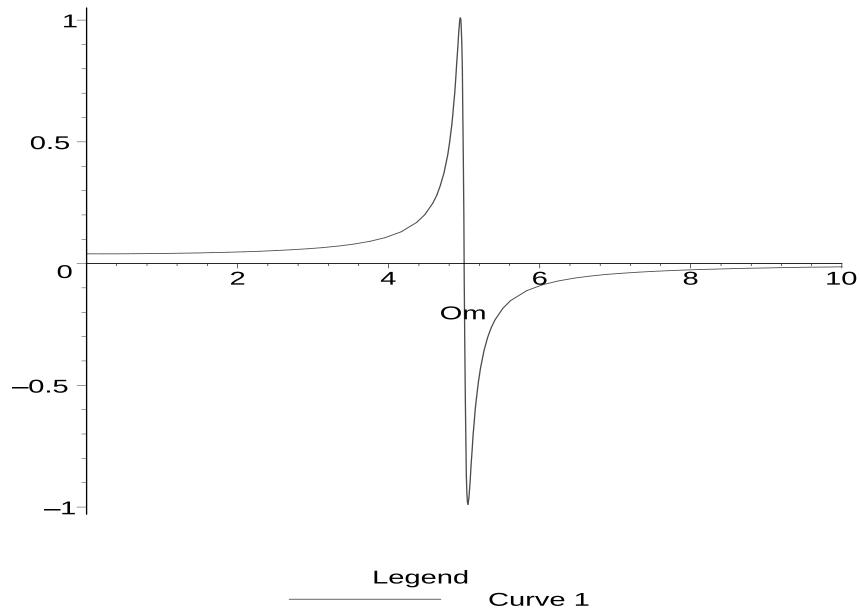
```
> A:=Om->(k-m*Om^2)/((k-m*Om^2)^2+(b*Om)^2);
```

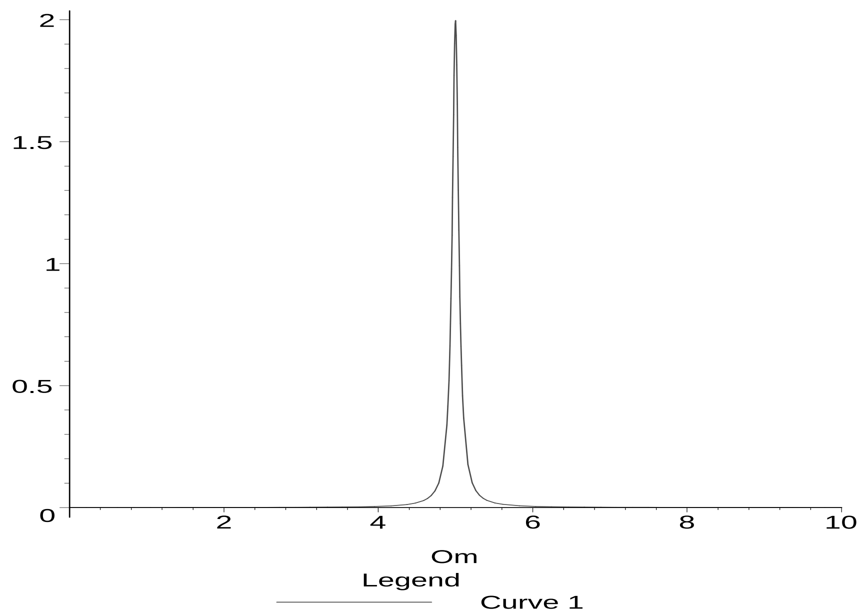
$$A := Om \to \frac{k - m\, Om^2}{(k - m\, Om^2)^2 + b^2\, Om^2}$$

```
> B:=Om->b*Om/((k-m*Om^2)^2+(b*Om)^2);
```

$$B := Om \to \frac{b\, Om}{(k - m\, Om^2)^2 + b^2\, Om^2}$$

```
> plot(A(Om), Om = 0..2*evalf(Om0));
> plot(B(Om), Om = 0..2*evalf(Om0));
```






```
> Om1=evalf(Om0);
```

$$Om1 = 5.$$

3 Problem 4.5.5

Find the general solution of

$$z'' + z' - z = 0 \ .$$

Solution:

$$\begin{array}{rcl} -z & = & -Ae^{rt} \\ + & & + \\ z' & = & Are^{rt} \\ + & & + \\ z'' & = & Ar^2e^{rt} \\ = & & = \\ 0 & = & A(r^2 + r - 1)e^{rt} \end{array}$$

Then, we factor

$$r^2 + r - 1 \Rightarrow r_\pm = \frac{-1 \pm \sqrt{5}}{2}$$

so that the general solution is

$$\begin{aligned} y(t) &= Ae^{r_+ t} + Be^{r_- t} \\ &= Ae^{\frac{-1+\sqrt{5}}{2}t} + Be^{\frac{-1-\sqrt{5}}{2}t} \end{aligned}$$

4 Problem 4.5.11

Find the general solution of

$$4w'' + 20w' + 25w = 0 \ .$$

Solution:

$$\begin{aligned} 4w &= 4Ae^{rt} \\ + & \quad + \\ 20w' &= 20rAe^{rt} \\ + & \quad + \\ 25w'' &= 25r^2Ae^{rt} \\ = & \quad = \\ 0 &= A(25r^2 + 20r + 4)e^{rt} \end{aligned}$$

Then, we factor

$$25r^2 + 20r + 4 = 0 \Rightarrow r = -\frac{2}{5} \ , \ \text{double}$$

so that the general solution is

$$\begin{aligned} w(t) &= Ae^{rt} + Bte^{rt} \\ &= Ae^{-\frac{2}{5}t} + Bte^{-\frac{2}{5}t} \end{aligned}$$

5 Problem 4.5.17

Solve the IVP

$$z'' - 2z' - 2z = 0 \ , \ z(0) = 0 \ , \ z'(0) = 3 \ .$$

Solution:

$$\begin{array}{rcl} -2z & = & -2Ae^{rt} \\ + & & + \\ -2z' & = & -2Are^{rt} \\ + & & + \\ z'' & = & Ar^2e^{rt} \\ = & & = \\ 0 & = & A(r^2-2r-2)e^{rt} \end{array}$$

Then, we factor

$$r^2 + r - 1 \Rightarrow r_\pm = 1 \pm \sqrt{3}$$

so that the general solution is

$$\begin{aligned} z(t) &= Ae^{r_+t} + Be^{r_-t} \\ &= Ae^{(1+\sqrt{3})t} + Be^{(1-\sqrt{3})t} \end{aligned}$$

Then

$$z(0) = 0 = A + B \ , \ z'(0) = 3 = A(1+\sqrt{3}) + B(1-\sqrt{3})$$

so

$$A + B = 0 \ , \ A - B = \sqrt{3} \Rightarrow A = -B = \frac{\sqrt{3}}{2}$$

giving for the solution:

$$z(t) = \frac{\sqrt{3}}{2}\left(e^{(1+\sqrt{3})t} - e^{(1-\sqrt{3})t}\right)$$