Solutions, 316-XXVI

April 30, 2003

26(4/26) Near-linear systems and critical points
12.3(1*,3,5*,7,9,11,13*,15*)
CAUTION: there may be errors!!!

1 Problem 12.3.1

Show that the given system is almost linear near the origin and discuss the type and stability of the critical point at the origin

$$\begin{aligned}\frac{dx}{dt} &= 3x + 2y - y^2 \\ \frac{dy}{dt} &= -2x - 2y + xy\end{aligned}$$

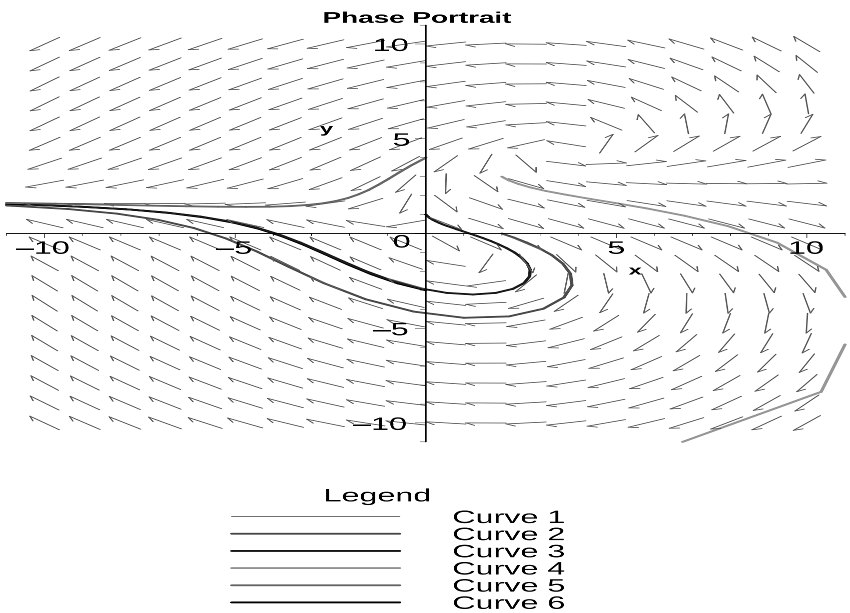
Solution:
Near the origin the nonlinear terms can be neglected. The critical point of the resulting linear system is a saddle. For such points the linearized system correctly characterizes the critical point:

$$\begin{vmatrix} 3-r & 2 \\ -2 & -2-r \end{vmatrix} = 0 \Rightarrow r^2 - r - 2 = (r-2)(r+1) = 0$$

i.e. $r_1 = 2$ and $r_2 = -1$. Therefore the origin is an unstable saddle point.

```
> restart:
> with(DEtools): with(plots): with(plottools):
> Digits := 10:
```

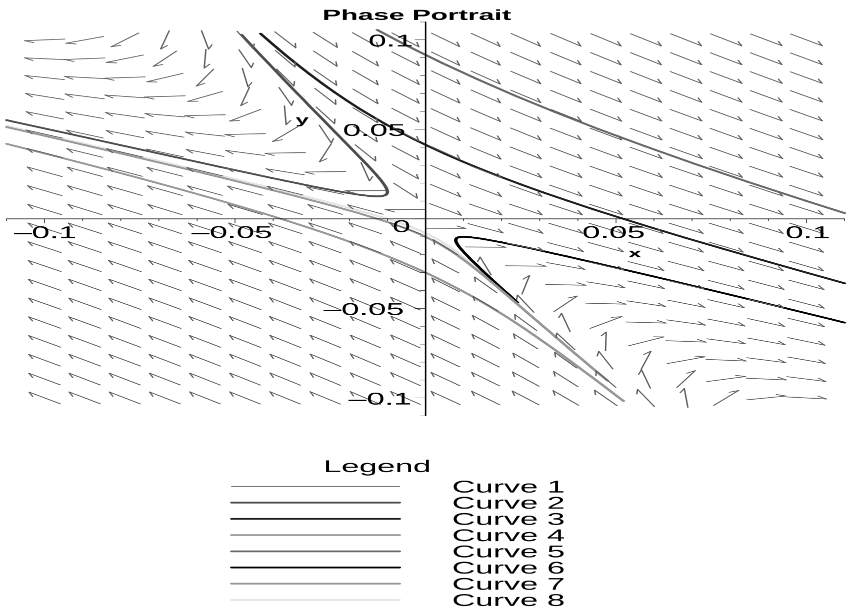
```
> plot1 :=
> DEplot([diff(x(t),t)=3*x(t)+2*y(t)-y(t)^2,diff(y(t),t)=-2*x(t)-2*y(t)+
> x(t)*y(t)], [x(t),y(t)], t=-0..2, scene=[x(t),y(t)],
> [[x(0)=2,y(0)=0],[x(0)=0,y(0)=4],[x(0) = 2,y(0)=3],[x(0) =
> 0,y(0)=1],[x(0)=0,y(0)=-3]], x=-10..10, y=-10..10,
> linecolor=[red,blue,green,magenta,black], arrows=SMALL, method=rkf45,
> stepsize=0.05):
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
> BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD,
10]);
```



This may not seem like it has a saddle point at the origin, but it is there if we look closer!

```
> plot1 :=
> DEplot([diff(x(t),t)=3*x(t)+2*y(t)-y(t)^2,diff(y(t),t)=-2*x(t)-2*y(t)+
> x(t)*y(t)], [x(t),y(t)], t=-2..2, scene=[x(t),y(t)],
> [[x(0)=-.01,y(0)=.015],[x(0)=0.01,y(0)=-.01],[x(0) =
> .1,y(0)=-.03],[x(0) =
> 0,y(0)=-.01],[x(0)=-.01,y(0)=0],[x(0)=.1,y(0)=.01],[x(0)=0,y(0)=-.03]]
> , x=-.1..0.1, y=-.1..0.1,
> linecolor=[red,blue,green,magenta,black,green,yellow], arrows=SMALL,
> method=rkf45, stepsize=0.05):
```

```
>  display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
>  BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD,
10]);
```



2 Problem 12.3.5

Show that the given system is almost linear near the origin and discuss the type and stability of the critical point at the origin

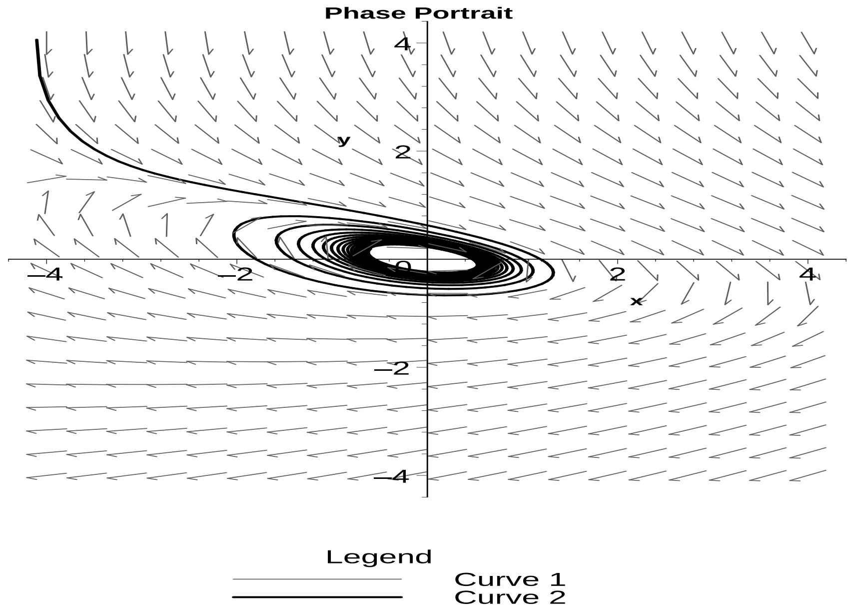
$$\begin{aligned} \frac{dx}{dt} &= x + 5y - y^2 \\ \frac{dy}{dt} &= -x - y - y^2 \end{aligned}$$

Solution:
Near the origin the nonlinear terms can be neglected. The critical point of the resulting linear system is a center. For such points the linearized system may not correctly characterize the critical point: the nonlinear system may possess a spiral instead of a center and the spiral can be stable or unstable.

$$\begin{vmatrix} 1-r & 5 \\ -1 & -1-r \end{vmatrix} = 0 \Rightarrow r^2 + 4 = (r - 2i)(r + 1i) = 0$$

i.e. $r_1 = 2i$ and $r_2 = -2i$. Therefore the origin is an center.

```
> plot1 :=
> DEplot([diff(x(t),t)=x(t)+5*y(t)-y(t)^2,diff(y(t),t)=-x(t)-y(t)-y(t)^2
> ], [x(t),y(t)], t=-10..50, scene=[x(t),y(t)], [[x(0)=1,y(0)=0]],
> x=-4..4, y=-4..4, linecolor=[black], arrows=SMALL, method=rkf45,
> stepsize=0.05):
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
> BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD,
10]);
```



We see that the origin is in reality an asymptotically stable spiral, not a neutrally stable center.

3 Problem 12.3.13

Find all the critical point of the given system, discuss the type and stability of each critical point and sketch the phase plane diagram in the vicinity of each critical point

$$\frac{dx}{dt} = 1 - xy$$
$$\frac{dy}{dt} = x - y^3$$

Solution:
We solve for the critical points:

$$xy = 1 \Rightarrow x = \frac{1}{y} \ ,$$

then

$$\frac{1}{y} - y^3 = 0 \Rightarrow y^4 = 1 \Rightarrow y = \pm 1 \text{ and } x = \pm 1 \ .$$

The linearization near the critical point (x_i, y_i) is governed by the matrix

$$\left| \begin{array}{cc} F_x & F_y \\ G_x & G_y \end{array} \right|_{(x_i,y_i)} = \left| \begin{array}{cc} -y_i & -x_i \\ 1 & -3y_i^2 \end{array} \right|$$

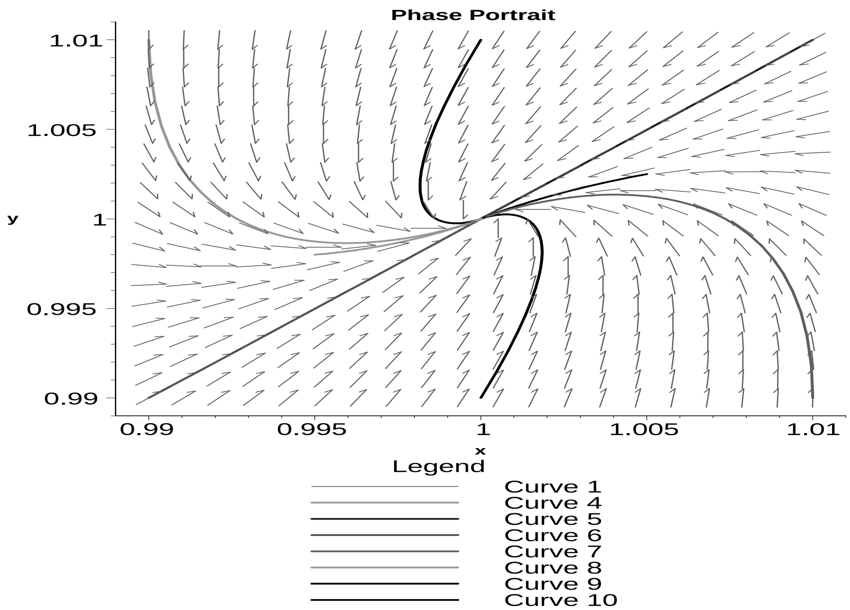
Near $(x_1, y_1) = (1, 1)$ we have

$$\left| \begin{array}{cc} -y_1 - r & -x_1 \\ 1 & -3y_1^2 - r \end{array} \right| = \left| \begin{array}{cc} -1 - r & -1 \\ 1 & -3 - r \end{array} \right| = r^2 + 4r + 4 = (r+2)^2 = 0 \ ,$$

so that the system has a stable 1-tangent node at $(1, 1)$. The nonlinear system does not necessarily have a 1-tangent node when the corresponding linear system does. In reality it could be a stable spiral or a stable improper node. As it turns out, the actual structure is that of a stable improper 2-tangent node.

```
> plot1 := DEplot([diff(x(t),t)=1-x(t)*y(t),diff(y(t),t)=x(t)-y(t)^3],
> [x(t),y(t)], t=0..40, scene=[x(t),y(t)],
> [[x(0)=1.01,y(0)=.99],[x(0)=.99,y(0)=.99],[x(0)=1.01,y(0)=1.01],[x(0)=
> .99,y(0)=1.01],[x(0)=1.01,y(0)=1.02],[x(0)=1.01,y(0)=1.03],[x(0)=1.005
> ,y(0)=1.0025],[x(0)=0.995,y(0)=0.998],[x(0)=1.,y(0)=1.01],[x(0)=1.0,y(
> 0)=.99]], x=.99..1.01, y=.99..1.01,
> linecolor=[black,magenta,green,blue,red,orange,green,black,black,black
> ], arrows=SMALL, method=rkf45, stepsize=0.05):
```

```
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
> BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD,
10]);
```



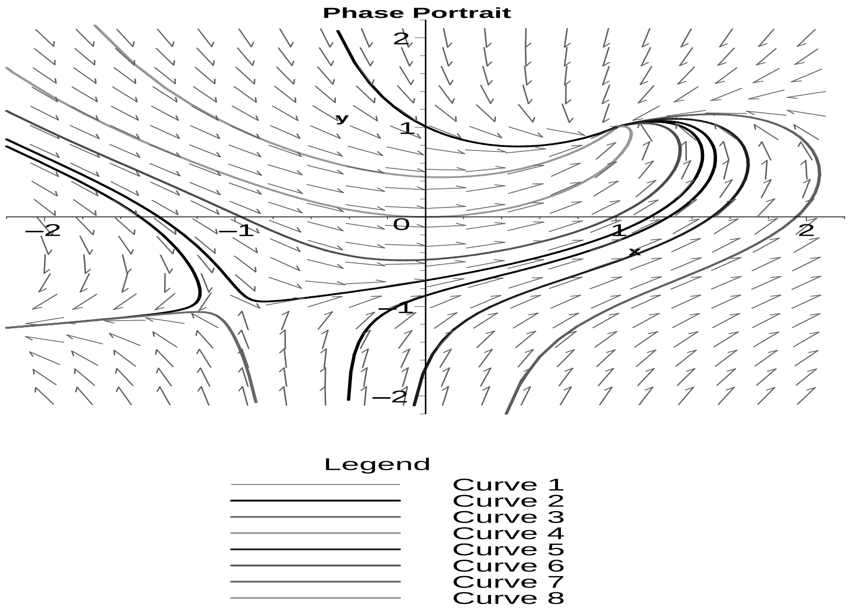
Near $(x_1, y_1) = (-1, -1)$ we have

$$\begin{vmatrix} 1-r & 1 \\ 1 & -3-r \end{vmatrix} = \begin{vmatrix} 1-r & 1 \\ 1 & -3-r \end{vmatrix} = r^2 + 2r - 4 = (r+1)^2 - 5 = 0\ ,$$

so that $r_1 = -1 + \sqrt{5} > 0$ while $r_2 = -1 - \sqrt{5} < 0$ and the linearized system at $(-1, -1)$ has an unstable saddle point there. Since saddles are found correctly by linearization, we conclude that the nonlinear stystem also has a saddle there.

```
> plot1 := DEplot([diff(x(t),t)=1-x(t)*y(t),diff(y(t),t)=x(t)-y(t)^3],
> [x(t),y(t)], t=-5..20, scene=[x(t),y(t)],
> [[x(0)=0,y(0)=0],[x(0)=-1.1,y(0)=-1.1],[x(0)=-1.4,y(0)=0],[x(0)=0,y(0)
> =1.01],[x(0)=-1.5,y(0)=0],[x(0)=0,y(0)=-0.88],[x(0)=-1.005,y(0)=1.0025
> ],[x(0)=0.995,y(0)=-0.998],[x(0)=1.,y(0)=0.01],[x(0)=1.0,y(0)=-.5]],
> x=-2..2, y=-2..2,
> linecolor=[black,magenta,green,blue,red,orange,green,black,black,black
> ], arrows=SMALL, method=rkf45, stepsize=0.05):
```

```
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
> BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD,
10]);
```



Near $(x_1, y_1) = (-1, -1)$ we have

4 Problem 12.3.15

Find all the critical point of the given system, discuss the type and stability of each critical point and sketch the phase plane diagram in the vicinity of each critical point

$$
\begin{aligned}
\frac{dx}{dt} &= 6x - 2x^2 - xy \\
\frac{dy}{dt} &= 6y - 2y^2 - xy
\end{aligned}
$$

Solution:

Critical points are found at $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (0,3)$, $(x_3, y_3) = (3,0)$, $(x_4, y_4) = (2,2)$. The linearization near the critical point (x_i, y_i) is governed by the matrix

$$
J(x_i, y_i) := \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_{(x_i,y_i)} = \begin{vmatrix} 6 - 4x_i - y_i & -x_i \\ -y_i & 6 - 4 * y_i - x_i \end{vmatrix}
$$

1. Near $(x_1, y_1) := (0,0)$ we have

$$
\begin{vmatrix} 6 - r & 0 \\ 0 & 6 - r \end{vmatrix} = (r-6)^2 = 0 \ ,
$$

   so that $r_1 = r_2 = 6$ and the linearized system at $(0,0)$ has an unstable star-node there. (equal eigenvalues but there is an eigenvector to each). The character of star nodes is also sensitive to perturbations, and the nonlinear system does not necessarily have one (it could have a stable node). Here it is preserved (i.e. the nonlinear system appears to have a real star at the origin.

2. Near $(x_2, y_2) := (3,0)$ we have

$$
\begin{vmatrix} -6 - r & -3 \\ 0 & 3 - r \end{vmatrix} = (r+6)(r-3) = 0 \ ,
$$

   so that $r_1 = 3$ and $r_2 = -6$ and the linearized system at $(0,3)$ has an unstable saddle there.

3. Near $(x_3, y_3) := (0, 3)$ we have

$$\begin{vmatrix} 3-r & 0 \\ -3 & -6-r \end{vmatrix} = (r+6)(r-3) = 0 \ ,$$

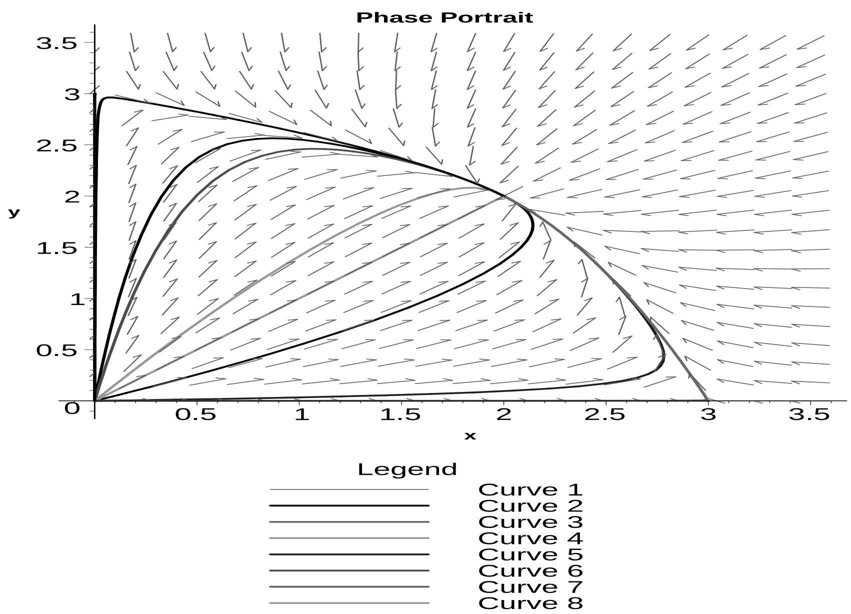
so that $r_1 = 3$ and $r_2 = -6$ and the linearized system at $(0, 3)$ has an unstable saddle there.

4. Near $(x_3, y_3) := (0, 3)$ we have

$$\begin{vmatrix} -4-r & -2 \\ -2 & -4-r \end{vmatrix} = (r+4)^2 - 4 = 0 \ ,$$

so that $r_1 = -2$ and $r_2 = -6$ and the linearized system at $(2, 2)$ has a stable improper node.

```
> plot1 :=
> DEplot([diff(x(t),t)=6*x(t)-2*x(t)^2-x(t)*y(t),diff(y(t),t)=6*y(t)-2*y
> (t)^2-x(t)*y(t)], [x(t),y(t)], t=-5..20, scene=[x(t),y(t)],
> [[x(0)=0,y(0)=0.1],[x(0)=.01,y(0)=.1],[x(0)=.5,y(0)=.5],[x(0)=.2,y(0)=
> .1],[x(0)=.3,y(0)=0],[x(0)=0.5,y(0)=2.01],[x(0)=2.05,y(0)=.1],[x(0)=0.
> 005,y(0)=0.008],[x(0)=2.5,y(0)=0.001],[x(0)=0.01,y(0)=2.5]],
x=0..3.5,
> y=0..3.5,
> linecolor=[black,magenta,green,blue,red,orange,green,black,black,black
> ], arrows=SMALL, method=rkf45, stepsize=0.05):
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
> BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD,
10]);
```


Phase Portrait
3.5
3
2.5
2
1.5
1
0.5
0
y
0.5
1
1.5
2
2.5
3
3.5
x
Legend
Curve 1
Curve 2
Curve 3
Curve 4
Curve 5
Curve 6
Curve 7
Curve 8