Solutions, 316-XXIV

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24(4/17) Real equal eigenvalues; mass-spring systems and forcing
9.5[35*,36], 9.6[19*,20], 9.7[7*,9,11,13*,16]
CAUTION: there may be errors!!!

1 Problem 9.5.35

1. Show that the matrix
$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$$
has the repeated eigenvalue $r = -1$ and that all the eigenvectors are of the form $\mathbf{u} = a\text{col}(1, 2)$.

2. Use the result of part (1) to obtain a nontrivial solution $\mathbf{x}_1(t)$ to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

3. To obtain a second linearly independent solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, try $\mathbf{x}_2(t) = te^{-t}\mathbf{u}_1 + e^{-t}\mathbf{u}_2$. [*Hint:* Substitute $\mathbf{x}_2$ into the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and derive the relations
$$(\mathbf{A} + \mathbf{I})\,\mathbf{u}_1 = 0\ ,\ (\mathbf{A} + \mathbf{I})\,\mathbf{u}_2 = \mathbf{u}_1\ .$$
Since $\mathbf{u}_1$ must be an eigenvector, set $\mathbf{u}_1 = \text{col}(1, 2)$ and solve for $\mathbf{u}_2$.]

4. What is $(\mathbf{A} + \mathbf{I})^2\,\mathbf{u}_2$?

Solution:

1. Eigenvalues:

$$\begin{vmatrix} 1-r & -1 \\ 4 & -3-r \end{vmatrix} = (r+3)(r-1)+4 = r^2+2r+1 = (r+1)^2 = 0 \ ,$$

so that $r=-1$ is a double eigenvalue. The only eigenvector associated with this eigenvalue is

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1-r \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ .$$

2. A solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is found as

$$\mathbf{x}_1(t) = e^{-t}\mathbf{u}_1 = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \ .$$

3. Try now a second solution in the form

$$\mathbf{x}_2(t) = te^{-t}\mathbf{u}_1 + e^{-t}\mathbf{u}_2 = te^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-t}\begin{pmatrix} a \\ b \end{pmatrix} \ .$$

Substituting into the system we find

$$\begin{aligned} \frac{d\mathbf{x}_2}{dt} &= (1-t)e^{-t}\mathbf{u}_1 - e^{-t}\mathbf{u}_2 \\ \mathbf{A}\mathbf{x} &= -te^{-t}\mathbf{u}_1 + e^{-t}A\mathbf{u}_2 \end{aligned}$$

and setting these equal we find that $\mathbf{u}_2$ satisfies:

$$A\mathbf{u}_2 = (-1)\mathbf{u}_2 + \mathbf{u}_1$$

or

$$(A-(-1)I)\mathbf{u}_2 = \mathbf{u}_1$$

i.e.

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ .$$

which is an Underdetermined system (two equations are the same), with a solution

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \ ,$$

while any multiple of $\mathbf{u}_1$ can be added to it to give another solution. The general solution is then

$$\mathbf{x}_{gen}(t) = C_1 e^{-t}\mathbf{u}_1 + C_2 e^{-t}\left(\mathbf{u}_2 + t\mathbf{u}_1\right) \ .$$

4. Here we have

$$(A-(-1)I)^2\mathbf{u}_2=(A-(-1)I)\mathbf{u}_1=0\ .$$

2 Problem 9.6.19

For the coupled mass-spring system governed by the system

$$\begin{aligned} m_1x_1'' &= -k_1x_1+k_2\,(x_2-x_1) \\ m_2x_2'' &= -k_2\,(x_2-x_1)-k_3x_2 \end{aligned}$$

assume $m_1=m_2=1kg$, $k_1=k_2=2N/m$ and $k_3=3N/m$. Determine the normal frequencies for this coupled mass-spring system.
(*Hint:* substitute $x_1=ae^{i\omega t}$, $x_2=be^{i\omega t}$ to arrive at a system of two equations in the unknown coefficients a, b; the system involves ω^2 in its coefficients. This system has the trivial solution $a=b=0$; to find a nontrivial solution the determinant of the coefficients of this system must vanish. Find the value(s) of ω^2 that make the determinant vanish. Their square roots should be real and they give the normal frequencies of the system.)
Solution:
Substituting $x_1=ae^{i\omega t}$ and $x_2=be^{i\omega t}$ (and dividing both sides by $e^{i\omega t}$) we find the system

$$\begin{aligned} -m_1a\omega^2 &= -k_1a+k_2\,(b-a) \\ -m_2b\omega^2 &= -k_2\,(b-a)-k_3b \end{aligned}$$

or

$$\begin{pmatrix} m_1\omega^2-(k_1+k_2) & k_2 \\ k_2 & m_2\omega^2-(k_2+k_3) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} \omega^2-4 & 2 \\ 2 & \omega^2-5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=0$$

which has a nontrivial solution provided the determinant of the coefficient matrix vanishes:

$$(\omega^2-4)(\omega^2-5)-4=0\Rightarrow\omega^4-9\omega^2+16=0\Rightarrow\omega^2=\frac{9\pm\sqrt{17}}{2}\ ,$$

so that the characteristic frequencies of the system are (multiply by 2π to get angular frequencies):

$$\omega_1=\sqrt{\frac{9+\sqrt{17}}{2}}\ ,\ \omega_2=\sqrt{\frac{9-\sqrt{17}}{2}}\ .$$

3 Problem 9.7.7

Determine the form of the particular solution for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$ by the method of undetermined coefficients if $\mathbf{A}$ and $\mathbf{f}(t)$ are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} , \mathbf{f}(t) = \begin{bmatrix} \sin 3t \\ t \end{bmatrix} .$$

Solution:
We need the homogeneous solution first:

$$r^2 - 2 = 0 \Rightarrow r_1 = \sqrt{2} \ , \ r_2 = -\sqrt{2}$$

with eigenvectors:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \ , \ \mathbf{u}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} .$$

Then

$$\mathbf{x}_{hom}(t) = C_1 e^{\sqrt{2}t}\mathbf{u}_1 + C_2 e^{-\sqrt{2}t}\mathbf{u}_2 .$$

Since none of the forcing terms are resonant, the particular solution will have the form

$$\mathbf{x}_{part}(t) = \begin{pmatrix} A_1 \cos t + B_1 \sin t + C_1 + D_1 t \\ A_2 \cos t + B_2 \sin t + C_2 + D_2 t \end{pmatrix}$$

and substituting into the system will give equations for determining the unknown coefficients $A_1, \ldots, D_2$.

4 Problem 9.7.13

Determine the general solution for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$ by the method of variation of parameters if $\mathbf{A}$ and $\mathbf{f}(t)$ are given by

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}, \mathbf{f}(t) = \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix} .$$

(**Instructions:**
Use the formula:

$$\mathbf{x}_g en(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)ds \; .$$

where $\mathbf{X}(t)$ is a fundamental matrix for the homogeneous system and $\mathbf{c}$ a constant vector.
NOTE: To find $\mathbf{X}^{-1}(s)$ you can use the formula given in class, but there is a **catch**: as mentioned in class,

$$\mathbf{X}(t-s) = \mathbf{X}(t)\mathbf{X}(-s) \; .$$

However, this means

$$\mathbf{X}(t-t) = \mathbf{X}(0) = \mathbf{X}(t)\mathbf{X}(-t)$$

so that

$$\mathbf{X}(-t) = \mathbf{X}^{-1}(t)\mathbf{X}(0)$$

i.e.

$$\mathbf{X}^{-1}(t) = \mathbf{X}(-t)\mathbf{X}^{-1}(0) \; .$$

In class I was assuming (but did not mention it!) that

$$\mathbf{X}(0) = I$$

which is not true in general. Also, it turns out that these matrices commute, i.e.

$$\mathbf{X}(t)\mathbf{X}(-t) = \mathbf{X}(-t)\mathbf{X}(t) = \mathbf{X}(0)$$

and

$$\mathbf{X}^{-1}(t) = \mathbf{X}(-t)\mathbf{X}^{-1}(0) = \mathbf{X}^{-1}(0)\mathbf{X}(-t)$$

so you can multiply them in any order you prefer!)
Solution:

1. Eigenvalues:

$$\left|\begin{array}{cc} 2-r & 1 \\ -3 & -2-r \end{array}\right| = 0 \ \Rightarrow \ r^2 - 1 = 0$$

so that

$$r_1 = 1 \quad , \quad r_2 = -1 \ .$$

2. Eigenvectors:

$$\mathbf{v}_1 = \left(\begin{array}{c} 1 \\ r_1 - 2 \end{array}\right) = \left(\begin{array}{c} 1 \\ -1 \end{array}\right) \ , \ \mathbf{v}_2 = \left(\begin{array}{c} 1 \\ r_2 - 2 \end{array}\right) = \left(\begin{array}{c} 1 \\ -3 \end{array}\right) \ .$$

3. Homogeneous solution:

$$\mathbf{x}_{hom} = C_1 e^t \left(\begin{array}{c} 1 \\ -1 \end{array}\right) + C_2 e^{-t} \left(\begin{array}{c} 1 \\ -3 \end{array}\right) \ .$$

4. Fundamental matrix:

$$bf X(t) = \left(\begin{array}{cc} e^t & e^{-t} \\ -e^t & -3e^{-t} \end{array}\right) \ .$$

5. Determinant of fundamental matrix (Wronskian):

$$\left|\begin{array}{cc} e^t & e^{-t} \\ -e^t & -3e^{-t} \end{array}\right| = -3 + 1 = -2 \ .$$

6. Inverse of fundamental matrix:

$$\mathbf{X}^{-1}(t) = \frac{-1}{2} \left(\begin{array}{cc} -3e^{-t} & -e^{-t} \\ e^t & e^t \end{array}\right) \ .$$

7. Integral:

$$\begin{aligned} \int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s) ds &= \frac{-1}{2} \int_0^t \left(\begin{array}{cc} -3e^{-s} & -e^{-s} \\ e^s & e^s \end{array}\right) \left(\begin{array}{c} 2e^s \\ 4e^s \end{array}\right) ds \\ &= \int_0^t \left(\begin{array}{c} 5 \\ -3e^{2s} \end{array}\right) ds \\ &= \left(\begin{array}{c} 5t \\ -\frac{3}{2}e^{2t} + \frac{3}{2} \end{array}\right) \end{aligned}$$

8. Particular solution:

$$\mathbf{X}(t)\int_{t_0}^{t}\mathbf{X}^{-1}(s)\mathbf{f}(s)ds = \begin{pmatrix} e^t & e^{-t} \\ -e^t & -3e^{-t} \end{pmatrix}\begin{pmatrix} 5t \\ -\frac{3}{2}e^{2t}+\frac{3}{2} \end{pmatrix}$$
$$= \begin{pmatrix} 5te^t - \frac{3}{2}e^t + \frac{3}{2}e^{-t} \\ -5te^t + \frac{9}{2}e^t - \frac{9}{2}e^{-t} \end{pmatrix}$$

9. General solution:

$$\mathbf{x}_{gen} = C_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 5te^t - \frac{3}{2}e^t + \frac{3}{2}e^{-t} \\ -5te^t + \frac{9}{2}e^t - \frac{9}{2}e^{-t} \end{pmatrix}$$

Note: this answer is equivalent to the answer in the book. They differ by a homogeneous solution (I did not simplify here to make it obvious how I got everything!).